

# COLLAPSING 4-MANIFOLDS UNDER A LOWER CURVATURE BOUND

TAKAO YAMAGUCHI

*Dedicated to Professor Katsuhiko Shiohama on his Sixtieth birthday*

**ABSTRACT.** In this paper we describe the topology of 4-dimensional closed orientable Riemannian manifolds with a uniform lower bound of sectional curvature and with a uniform upper bound of diameter which collapse to metric spaces of lower dimensions. This enables us to understand the set of homeomorphism classes of closed orientable 4-manifolds with those geometric bounds on curvature and diameter. In the course of the proof of the above results, we obtain the soul theorem for 4-dimensional complete noncompact Alexandrov spaces with nonnegative curvature. A metric classification for 3-dimensional complete Alexandrov spaces with nonnegative curvature is also given.

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## 0. INTRODUCTION

The study of the Gromov-Hausdorff convergence of Riemannian manifolds has been a significant branch in differential geometry. In the study of convergence or collapsing of Riemannian manifolds, one usually considers a curvature bound of Riemannian manifolds. Our main concern is the study of the collapsing phenomena of Riemannian manifolds under a uniform lower bound on the sectional curvature  $K$ . When the absolute value  $|K|$  is uniformly bounded, Cheeger, Fukaya and Gromov [10] developed a general theory of collapsing, where the collapsing phenomena were described in terms of the generalized group actions by nilpotent groups, called  $N$ -structures. It should be noted that these actions are not permitted to have fixed points. Now if we turn attention to the case when  $K$  has only a uniform lower bound, we recognize several kinds of different collapsing phenomena in this new situation:

- *Any* effective action on a compact manifold by a compact connected Lie group of positive dimension causes a collapsing under a lower curvature bound ([43]);
- There is a phenomenon of curvature-concentration, for instance in the convergence of surfaces to a singular surface with conical singularities, under a lower curvature bound.

Thus the study of collapsing of Riemannian manifolds under a lower curvature bound will enable us to understand a wider class of collapsing phenomena than the case of absolute bound on  $K$ .

In spite of several studies of collapsing or convergence under a lower sectional curvature bound, the general collapsing structure is still unclear. Recently in [43], we have classified all the collapsing phenomena of 3-manifolds under a lower curvature bound. In the present paper, we discuss and determine the collapsing of 4-manifolds under a lower curvature bound. Note that  $S^4$ ,  $\pm\mathbb{C}P^2$ ,  $S^2 \times S^2$  and arbitrary connected sum of them all admit nontrivial  $S^1$ -actions, and therefore can collapse under  $K \geq -1$  while no simply connected closed 4-manifold can collapse under  $|K| \leq 1$  because of nonzero Euler characteristics ([9], [12]).

For a given  $D > 0$ , let  $\mathcal{M}(4, D)$  denote the family of closed orientable Riemannian 4-manifolds  $M$  with a lower sectional curvature bound  $K \geq -1$  and an upper diameter bound  $\text{diam}(M) \leq D$ . Since  $\mathcal{M}(4, D)$  is precompact ([23]), it is quite natural to consider an infinite sequence  $M_i^4$  in  $\mathcal{M}(4, D)$  converging to a compact metric space  $X$  with respect to the Gromov-Hausdorff distance. Now it is well-known that  $X$  has the structure of Alexandrov space with curvature  $\geq -1$  and of dimension  $\leq 4$ . It is a quite important problem to reproduce the topology of  $M_i^4$  for sufficiently large  $i$  by using the information on the singularities of  $X$ . This enables us to understand the elements of a small neighborhood of  $X$  in  $\mathcal{M}(4, D)$  and therefore to describe the structure of  $\mathcal{M}(4, D)$  itself because of the precompactness.

In the case of  $\dim X = 4$  or  $\dim X = 0$ , we know some answers for this problem ([35], [20]). In this paper, we discuss the cases of  $1 \leq \dim X \leq 3$ . Our results can be simply stated as follows:

**Theorem 0.1.** *Suppose  $1 \leq \dim X \leq 3$ . Then  $M_i^4$  has a singular fibre structure over  $X$  in a generalized sense.*

Here the fibre type is constant along each strata of a stratification of  $X$ , but may change when the strata changes. In particular,  $M_i^4$  is a fibre bundle over  $X$  if  $X$  has no *essential singular points*.

To state our results more precisely and more explicitly, let us describe the collapsing structure in each case of the dimension of  $X$ .

We begin with the case of  $\dim X = 3$ .

**Theorem 0.2.** *Suppose that  $M_i^4$  collapses to a three-dimensional compact Alexandrov space  $X^3$  under  $K \geq -1$ . Then there exists a locally smooth, local  $S^1$ -action  $\psi_i$  on  $M_i^4$  such that the orbit space  $M_i^4/\psi_i$  is homeomorphic to  $X^3$ .*

*Remark 0.3.* In Theorem 0.2, one can obtain the information on the slice representations of the isotropy groups at the fixed points or the exceptional orbits of  $\psi_i$  from the information on the singularities of the

corresponding singular points of  $X$  as described below. See Theorems 7.1 and 8.1 for details:

- (1) We have a bijection  $\iota_i$  of the fixed point set  $F(\psi_i)$  to the union of  $\partial X^3$  and a subset of  $\text{Ext}(\text{int } X^3)$ , where  $\text{Ext}(\text{int } X^3)$ , the set of *extremal points* of  $\text{int } X^3$ , denotes the set of points  $p$  of  $\text{int } X^3$  whose spaces of directions,  $\Sigma_p$ , have diameters  $\leq \pi/2$ .
- (2) The singular locus  $S(\psi_i) \subset X$  of  $\psi_i$  is a quasigeodesic consisting of essential singular points of  $X^3$ .

*Remark 0.4.* In a course of the proof of Theorem 0.2, we prove that the limit space  $X^3$  is a topological manifold (Proposition 6.6).

In [16], Fintushel classified locally smooth  $S^1$ -actions on simply connected closed 4-manifolds. Applying this result to Theorem 0.2, we obtain

**Corollary 0.5.** *Under the hypothesis of Theorem 0.2, suppose that  $M_i^4$  is simply connected. Then  $M_i^4$  is homeomorphic to the connected sum*

$$(0.1) \quad S^4 \# k_i \mathbb{C}P^2 \# \ell_i(-\mathbb{C}P^2) \# m_i(S^2 \times S^2),$$

where  $k_i$ ,  $\ell_i$  and  $m_i$  have a definite upper bound in terms of the number of the extremal points of  $\text{int } X^3$  and the number, say  $\alpha(\partial X^3)$ , of the connected components of  $\partial X^3$ :

$$k_i + \ell_i + 2m_i \leq \# \text{Ext}(\text{int } X^3) + 2\alpha(\partial X^3) - 2.$$

Actually  $M_i^4$  is diffeomorphic to the connected sum (0.1) (see Remark 8.11).

Now we turn to the case of  $\dim X = 2$ . First we consider the case when  $X^2$  has no boundary. By definition, a closed 4-manifold is a Seifert  $T^2$ -bundle over  $X^2$  if there exists a surjective continuous map  $f : M^4 \rightarrow X^2$  such that any point  $p \in X^2$  has a neighborhood  $U$  admitting the following commutative diagram :

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\simeq} & (T^2 \times D^2)/\mathbb{Z}_m \\ f \downarrow & & \downarrow \pi \\ U & \xrightarrow{\simeq} & D^2/\mathbb{Z}_m, \end{array}$$

where  $(T^2 \times D^2)/\mathbb{Z}_m$  denotes a free diagonal  $\mathbb{Z}_m$ -quotient; the  $\mathbb{Z}_m$ -action is free on the  $T^2$ -factor and by rotation on the  $D^2$ -factor,  $p$  corresponds to the  $\mathbb{Z}_m$ -fixed point of  $D^2$  in the above diagram, and  $m$  is a positive integer called the multiplicity of the torus fibre  $f^{-1}(p)$ .

**Theorem 0.6.** *Suppose that  $M_i^4$  collapses to a two-dimensional compact Alexandrov space  $X^2$  without boundary under  $K \geq -1$ . Then  $M_i^4$  is homeomorphic to either an  $S^2$ -bundle over  $X^2$  or a Seifert  $T^2$ -bundle over  $X^2$ .*

Thus if the general fibre is a torus,  $M_i^4$  may have singular torus-fibres (multiple tori) near essential singular points of  $X^2$ , while in the sphere-fibre case,  $M_i^4$  is just a sphere-bundle even if  $X^2$  has serious singular points. It should be remarked that the multiplicity  $m$  of a multiple torus near a essential singular point  $p \in X$  can be estimated as  $m \leq 2\pi/L(\Sigma_p)$ , where  $L(\Sigma_p)$  denotes the length of  $\Sigma_p$ .

Next we consider the case when the compact Alexandrov surface  $X^2$  has nonempty boundary. In this case, we have a decomposition of  $M_i^4$  into two parts: one is a part of  $M_i^4$  near a compact domain of  $\text{int } X^2$  approximating  $\text{int } X^2$  which is either an  $S^2$ -bundle or a Seifert  $T^2$ -bundle (Theorem 0.6). The other is a part of  $M_i^4$  near  $\partial X^2$ , on which we can define a singular fibre structure as indicated in the theorem below.

**Theorem 0.7.** *Suppose that  $M_i^4$  collapses to a two-dimensional compact Alexandrov space  $X^2$  with boundary under  $K \geq -1$ . Then  $M_i^4$  is homeomorphic to a singular fibre space  $\mathcal{F}(X^2)$  over  $X^2$ .*

As a particular consequence of Theorem 0.7, there is a continuous surjective map  $f_i : M_i^4 \rightarrow X^2$  such that  $f_i$  restricted to  $\text{int } X^2$  is either an  $S^2$ -bundle or a Seifert  $T^2$ -bundle and  $f_i$  restricted to  $\partial X^2$  is a singular fibration whose fibres are ones of a point,  $S^1$ ,  $S^2$ , the real projective plane  $P^2$  and the Klein bottle  $K^2$ . The fiber types of  $f_i|_{\partial X^2}$  may change at extremal points  $\in \text{Ext}(X^2) \cap \partial X^2$ . (see Theorem 12.1 for details).

Finally we assume  $\dim X = 1$ . If  $X$  is a circle, we can apply the fibration theorem ([46]) to get a fibre bundle structure on  $M_i^4$  over  $X$ . More precisely,

**Theorem 0.8** ([20]). *Suppose that  $M_i^4$  collapses to a circle under  $K \geq -1$ . Then  $M_i^4$  fibers over  $S^1$  with fibre finitely covered by  $S^1 \times S^2$ ,  $T^3$ , a nilmanifold or a homotopy 3-sphere.*

Thus the essential case here is the case when  $X$  is isometric to a closed interval:

**Theorem 0.9.** *Suppose that  $M_i^4$  collapses to a one-dimensional closed interval under  $K \geq -1$ . Then  $M_i^4$  is homeomorphic to a gluing  $U_i^4 \cup V_i^4$  along their boundaries, where each of  $U_i^4$  and  $V_i^4$  is either a disk bundle over  $k$ -dimensional closed manifold  $N^k$  with  $0 \leq k \leq 3$ , or a gluing of two disk-bundles over  $k_j$ -dimensional closed manifolds  $Q^{k_j}$ ,  $j = 1, 2$ , with  $0 \leq k_j \leq 2$ , where  $N^k$  and  $Q^{k_j}$  have nonnegative Euler numbers, and if  $k = 3$ ,  $N^3$  is one of the closed 3-manifolds appearing as the fibre types in Theorem 0.8.*

*Therefore  $M_i^4$  is a gluing of at least two and at most four disk-bundles.*

In the case when  $X$  is a point, rescaling metric of  $M_i^4$  with the new diameter 1, we can reduce the problem to the cases of  $1 \leq \dim X \leq 4$ .

As a conclusion of the results in this paper together with the stability theorem ([35]) and the above remark, we have a description of the homeomorphism classes in  $\mathcal{M}(4, D)$  as follows:

**Corollary 0.10.** *For a given  $D > 0$ , there exist finitely many elements  $N_1^4, \dots, N_k^4$  of  $\mathcal{M}(4, D)$ , where  $k = k(D)$ , such that for any element  $M^4$  of  $\mathcal{M}(4, D)$  one of the following holds:*

- (1)  $M^4$  is homeomorphic to one of  $\{N_1^4, \dots, N_k^4\}$ ;
- (2)  $M^4$  is homeomorphic to a closed 4-manifold as described in Theorem 0.2, 0.6, 0.7, 0.8 or 0.9 with a suitable compact Alexandrov space  $X$  of  $1 \leq \dim X \leq 3$ .

In the proof of each result mentioned above, it is crucial to understand the topology of a small (but of a fixed radius) metric ball in a collapsed 4-manifold which is very close to a singular point of the limit space. As the important first step, we establish the rescaling technique in dimension 4 which makes it possible to pursue the topology of the metric ball. Under the rescaling of metrics, we obtain a new limit space, a complete noncompact Alexandrov space with nonnegative curvature and with dimension larger than that of the original limit space. This reduces the problem to the study of collapsing or convergence to a higher dimensional spaces (with nonnegative curvature). The second step is to analyze the new limit space in several aspects. Among them, we especially need to establish the generalized soul theorem (Theorem 2.6) for 4-dimensional complete open Alexandrov spaces with nonnegative curvature. In the third step, we actually determine the structure of collapsing to the new limit space, and obtain the topology of the small metric ball using an inductive procedure. Using this topological information, we define a local fibre structure caused by collapsing. Through some patching argument under the consideration of the singular point set of the limit space, we glue those locally defined fibre structures to obtain a globally defined fibre structure.

A similar strategy was used in [43]. However the present situation is more involved in several aspects like; a generalization of the rescaling argument in [43] (our approach seems to be possible to generalize to the general dimensions), the complexity of the singular sets in the limit spaces, the generalized soul theorem, classification of the collapses to the complete noncompact nonnegatively curved spaces with dimension less than four.

Our results in this paper suggest a possibility of the study of collapse of general  $n$ -dimensional Riemannian manifolds with a lower curvature bound by the following program:

**Assumption ( $A_k$ )** We already know the fibre structure of closed  $n$ -manifolds collapsing to spaces of dimension  $\geq k$ , where the orbit types of the singular fibres can be determined or estimated in terms of the

singularities of the corresponding singular points of the limit spaces (this is always satisfied for  $k = n$ ).

**Purpose** ( $P_{k-1}$ ) We want to determine the structure of closed  $n$ -manifolds collapsing to  $(k - 1)$ -dimensional spaces.

To realize ( $P_{k-1}$ ), in the first step, we obtain the topology of a small metric ball using the following procedure

- (a) a suitable rescaling of metrics;
- (b) the (metric) classification of the complete noncompact spaces of dimension  $\geq k$  and with nonnegative curvature by means of the geometric invariants of them (like souls, the dimensions of the ideal boundaries, the extremal point sets and the essential singular point sets);
- (c) the classification of the phenomena of collapsing to the spaces in (b) using ( $A_k$ );

In the second step, we first define a fibre structure on the small metric ball, and then do a gluing process under the presence of the singular point set of the limit space and realize the purpose ( $P_{k-1}$ ).

As stated above, this program has been carried out for  $n = 4$  in the present paper.

The organization of this paper is as follows: In Section 1, we provide some basic notions about Alexandrov spaces with curvature bounded below and the Gromov-Hausdorff convergence. We formulate a main result in this section, the fibration-capping theorem, which gives a fibre structure on a collapsed manifold in the case when the limit space has nonempty boundary. The proof of this result is postponed to Sections 18-21 (Part3), where we actually prove an equivariant version of it.

In Section 3, we prepare the basic results about  $S^1$ -actions on oriented 4-manifolds ([16]) providing the properties of singular loci and the equivariant classification of such actions in terms of weighted orbit data. We apply those results to obtain a topological classification of  $S^1$ -action on compact 4-manifolds with certain simple orbit data in terms of  $D^2$ -bundles over  $S^2$ . This result is useful to classify the collapsing phenomena of 4-manifolds to 3-dimensional complete noncompact spaces with nonnegative curvature.

In Section 4, we establish a key result to obtain a topology of a small metric ball of a collapsed 4-manifold close to a singular point of the limit space by generalizing a rescaling argument in [43]. This makes it possible to reduce the problem to the study of collapsing to higher dimensional limit spaces of nonnegative curvature.

The rescaling argument in Section 4 makes clear that the first step in the study of collapsing of 4-manifolds should be the case when the limit space is of dimension three. In Section 5, as a preliminary section, we establish two results about the properties of the essential singular point set of 3-dimensional Alexandrov spaces with curvature bounded

below: One is a characterization of certain essential singular point set in terms of quasigeodesics, and the other is about the existence of a metric collar neighborhood.

From Section 6 to Section 8, we analyze the collapsing phenomena of orientable 4-manifolds to 3-dimensional Alexandrov spaces. In Section 6, we concentrate on finding the collapsing structure on a small metric ball in a collapsed 4-manifold which is close to a small metric ball in the interior of the limit three-space. As the local collapsing structure, we construct an  $S^1$ -action on a small perturbation of the small metric ball extending the  $S^1$ -bundle structure on a regular part of the 4-manifold. In Section 7, we patch those locally defined  $S^1$ -actions to obtain a globally defined local  $S^1$ -action on a collapsed 4-manifold. In Section 8, we determine the structure of collapsing to a 3-dimensional Alexandrov space  $X^3$  with boundary using the collar neighborhood theorem established in Section 5 and the capping theorem, a generalization of the fibration theorem. We prove that a part of a collapsed 4-manifold close to the boundary  $\partial X^3$  is a  $D^2$ -bundle over  $\partial X^3$ , compatible with the  $S^1$ -bundle structure near  $\partial X^3$ . This provides a globally defined, local  $S^1$ -action on the 4-manifold.

Before proceeding to the study of collapsing of 4-manifolds to 2-dimensional Alexandrov spaces, in Section 9 we classify all the phenomena of collapsing of pointed complete 4-manifolds to a pointed complete noncompact 3-dimensional Alexandrov spaces with nonnegative curvature. Using  $S^1$ -actions with the study of the singular locus, we determine the topology of large metric balls in the 4-manifolds centered at the reference points. We prove that they are homeomorphic to either a disk-bundle or a gluing of two disk-bundles.

In Sections 10 and 11, we apply the classification result in Section 9 to obtain the topological information on a small metric ball in a 4-manifold collapsed to 2-dimensional space without boundary. Considering the new limit space under the rescaling of metrics, we come to know what kinds of collapsing phenomena really occur near a singular fibre. After those discussions together with the equivariant fibration theorem, we prove Theorem 0.6 putting the desired fibre structures.

In Section 12, we consider the case when the limit surface has nonempty boundary, and prove Theorem 0.7.

In Section 13, we classify all the phenomena of collapsing of pointed complete 4-manifolds to a pointed complete noncompact 2-dimensional Alexandrov spaces with nonnegative curvature using Theorems 0.6 and 0.7. Theorem 0.9 is also proved there.

In Sections 14–16 of Part 2, we prove the generalized soul theorem in dimension 4, and in Section 17 we classify all the 3-dimensional complete nonnegatively curved Alexandrov spaces with nonempty boundary.



In Sections 18–21 of Part 3, we give the proof of the equivariant fibration-capping theorem.

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## 1. ALEXANDROV SPACES AND THE GROMOV-HAUSDORFF CONVERGENCE

In this section, we present some results on Alexandrov spaces with curvature bounded below and on the Gromov-Hausdorff convergence which will be used in the subsequent sections.

**1. Alexandrov spaces.** We begin with some preliminary results on Alexandrov spaces. We refer to [6] for the basic materials and the details on Alexandrov spaces mentioned below.

Let  $X$  be a finite-dimensional complete Alexandrov space with curvature bounded below, say  $\geq \kappa$ . For any two point  $x$  and  $y$ , let  $xy$  denote a minimal geodesic joining  $x$  to  $y$ . For any geodesic triangle  $\Delta xyz$  in  $X$  with vertices  $x, y$  and  $z$ , if we denote by  $\tilde{\Delta}xyz$  a *comparison triangle* in the  $\kappa$ -plane  $M_\kappa^2$ , the simply connected complete surface with constant curvature  $\kappa$ , the natural map  $\Delta xyz \rightarrow \tilde{\Delta}xyz$  is non-expanding, where we assume that the perimeter of  $\Delta xyz$  is less than  $2\pi/\sqrt{\kappa}$  if  $\kappa > 0$ . This property is called the *Alexandrov convexity*.

The angle between the geodesics  $xy$  and  $yz$  in  $X$  is denoted by  $\angle xyz$ , and the corresponding angle of  $\tilde{\Delta}xyz$  by  $\tilde{\angle}xyz$ . It holds that  $\angle xyz \geq \tilde{\angle}xyz$ . We denote by  $\Sigma_p = \Sigma_p(X)$  the space of directions at  $p \in X$ , and by  $K_p = K_p(X)$  the tangent cone at  $p$  with vertex  $o_p$ , the Euclidean cone  $K(\Sigma_p)$  over  $\Sigma_p$ . It is known that  $\Sigma_p$  (resp.  $K_p$ ) is an Alexandrov space with curvature  $\geq 1$  (resp  $\geq 0$ ). More generally if  $\Sigma$  is a compact Alexandrov space with curvature  $\geq 1$ , then the Euclidean cone  $K(\Sigma)$  over  $\Sigma$  is a complete Alexandrov space with nonnegative curvature.

For a compact set  $A \subset X$  and  $p \in X - A$ , we denote by  $A' = A'_p$  the closed set of  $\Sigma_p$  consisting of all the directions of minimal geodesics from  $p$  to  $A$ .

Let  $k$  be the dimension of  $X$ , and  $\delta > 0$ . A system of  $k$  pairs of points,  $(a_i, b_i)_{i=1}^k$  is called an  $(k, \delta)$ -*strainer* at  $p \in X$  if it satisfies

$$\begin{aligned} \tilde{\angle} a_i p b_i &> \pi - \delta, & \tilde{\angle} a_i p a_j &> \pi/2 - \delta, \\ \tilde{\angle} b_i p b_j &> \pi/2 - \delta, & \tilde{\angle} a_i p b_j &> \pi/2 - \delta, \end{aligned}$$

for every  $i \neq j$ . The number  $\min \{d(a_i, p), d(b_i, p) \mid 1 \leq i \leq k\}$  is called the *length* of the strainer.

Let  $R_\delta(X)$  denote the set of  $(k, \delta)$ -strained points of  $X$ . Note that every point in  $R_\delta(X)$  has a small neighborhood almost isometric to

an open subset of  $\mathbb{R}^k$  for small  $\delta$ , where the almost isometry is given by  $f(\cdot) = (d(a_1, \cdot), \dots, d(a_k, \cdot))$ . In particular,  $R_\delta(X)$  is a Lipschitz  $k$ -manifold. The set  $S_\delta(X) := X - R_\delta(X)$  is called the  $\delta$ -singular set of  $X$ .

A point  $p$  of an Alexandrov space  $X$  is called an *extremal* point if  $\text{diam}(\Sigma_p) \leq \pi/2$ . This coincides with the case when an extremal subset in the sense of [38] is a point. The set of all extremal points of  $X$  is denoted  $\text{Ext}(X)$ . A point  $p$  of  $X$  is called an *essential* singular point if the radius of  $\Sigma_p$  satisfies  $\text{rad}(\Sigma_p) := \min_\xi \max_\eta \angle(\xi, \eta) \leq \pi/2$ , and the set of all essential singular points of  $X$  is denoted  $ES(X)$ . Notice that if a point  $p \in X$  is not an essential singular point, then  $\Sigma_p$  is homeomorphic to a sphere ([24], [38]), and a small metric ball around  $p$  is homeomorphic to  $\mathbb{R}^k$  ([35]).

We also say that  $p$  is a *topological* singular point if  $\Sigma_p$  is not homeomorphic to a sphere. If  $p$  is not a topological singular point, then it is called topologically regular.  $X$  is called *topologically regular* if any point of  $X$  is topologically regular. It is well known (cf.[15]) that the double suspension  $S^2(\Sigma^3)$  of the Poincare homology 3-sphere  $\Sigma^3$  is a topological sphere. Since  $S^2(\Sigma^3)$  carries a metric as an Alexandrov space of curvature  $\geq 1$ , one recognizes that a topological singular point of an Alexandrov space can be a manifold point at least in dimension  $\geq 5$ . The non-collapsing limit of a sequence of  $n$ -manifolds with a uniform lower sectional curvature bound is called *smoothable*. Recently in [29], V. Kapovitch has proved that a smoothable Alexandrov space is topologically regular. More precisely,

**Proposition 1.1** ([29]). *Let  $X^n$  be a smoothable Alexandrov space. Then for any  $p_0 \in X^n$ ,  $p_1 \in \Sigma_{p_0}X, \dots, p_i \in \Sigma_{p_{i-1}} \cdots \Sigma_{p_0}X$ , the iterated space of directions  $\Sigma_{p_i} \Sigma_{p_{i-1}} \cdots \Sigma_{p_0}X$  is a topological sphere.*

We call a point  $p_0$  of an Alexandrov space  $X^n$  *topologically nice* if it satisfies the conclusion of Proposition 1.1 for any  $p_1 \in \Sigma_{p_0}X, \dots, p_i \in \Sigma_{p_{i-1}} \cdots \Sigma_{p_0}X$ . We also call  $X^n$  *topologically nice* if any point of  $X^n$  is topologically nice. It is obvious that for  $n \leq 4$ , if  $X^n$  is topologically regular, then it is topologically nice. When  $m \geq 3$ ,  $S^m(\Sigma^3)$  is not topologically nice but topologically regular.

**2. Gromov-Hausdorff convergence.** A (not necessarily continuous) map  $\varphi : Y \rightarrow Z$  between metric spaces is called an  $\epsilon$ -approximation if

- (1)  $|d(x, y) - d(\varphi(x), \varphi(y))| < \epsilon$  for all  $x, y \in Y$ ,
- (2)  $\varphi(Y)$  is  $\epsilon$ -dense in  $Z$ .

The Gromov-Hausdorff distance  $d_{GH}(Y, Z)$  between  $Y$  and  $Z$  is defined to be the infimum of such  $\epsilon$  that there exist  $\epsilon$ -approximations  $Y \rightarrow Z$  and  $Z \rightarrow Y$ . The pointed Gromov-Hausdorff distance  $d_{p.GH}((Y, y), (Z, z))$

between pointed metric spaces  $(Y, y)$  and  $(Z, z)$  is defined as the infimum of such  $\epsilon$  that there exist  $\epsilon$ -approximations  $B(y, 1/\epsilon) \rightarrow B(z, 1/\epsilon)$  and  $B(z, 1/\epsilon) \rightarrow B(y, 1/\epsilon)$  sending  $y$  to  $z$  and  $z$  to  $y$  respectively.

In the study of collapsing of three-dimensional Riemannian manifolds with a lower curvature bound, the so-called fibration theorem ([46], etc.) has played one of fundamental roles ([43]). This theorem shows that if a complete Riemannian manifold  $M$  with a definite lower bound on sectional curvature is Gromov-Hausdorff close to a Riemannian manifold  $X$  of lower dimension without boundary (or more generally an Alexandrov space with only weak singularities ([47])), then  $M$  fibers over  $X$  with almost nonnegatively curved fibre. We generalize this result by considering  $X$  with nonempty boundary as follows :

Let  $X$  be a  $k$ -dimensional complete Alexandrov space with curvature  $\geq -1$ . Now we assume that  $X$  has nonempty boundary, and denote by  $D(X)$  the double of  $X$ , which is also an Alexandrov space with curvature  $\geq -1$  (see [35]). By definition,  $D(X) = X \cup X^*$  glued along their boundaries, where  $X^*$  is another copy of  $X$ .

A  $(k, \delta)$ -strainer  $\{(a_i, b_i)\}$  of  $D(X)$  at  $p \in X$  is called *admissible* if  $a_i \in X, b_j \in X$  for every  $1 \leq i \leq k, 1 \leq j \leq k-1$  (obviously,  $b_k \in X^*$  if  $p \in \partial X$  for instance). Let  $R_\delta^D(X)$  denote the set of points of  $X$  at which there are admissible  $(k, \delta)$ -strainers. This has the structure of a Lipschitz  $k$ -manifold with boundary. Note that every point of  $R_\delta^D(X) \cap \partial X$  has a small neighborhood in  $X$  almost isometric to an open subset of the half space  $\mathbb{R}_+^n$  for small  $\delta$ .

If  $Y$  is a closed domain of  $R_\delta^D(X)$ , then the  $\delta_D$ -strain radius of  $Y$ , denoted  $\delta_D\text{-str.rad}(Y)$ , is defined as the infimum of positive numbers  $\ell$  such that there exists an admissible  $(k, \delta)$ -strainer of length  $\geq \ell$  at every point  $p \in Y$ .

For a small  $\nu > 0$ , we put

$$Y_\nu := \{x \in Y \mid d(\partial X, x) \geq \nu\}.$$

We use the following special notations:

$$\partial_0 Y_\nu := Y_\nu \cap \{d_{\partial X} = \nu\}, \quad \text{int}_0 Y_\nu := Y_\nu - \partial_0 Y_\nu.$$

Let  $M^n$  be another  $n$ -dimensional complete Alexandrov space with curvature  $\geq -1$ . A surjective map  $f : M \rightarrow X$  is called an  $\epsilon$ -almost Lipschitz submersion if

- (1) it is an  $\epsilon$ -approximation;
- (2) for every  $p, q \in M$

$$\left| \frac{d(f(p), f(q))}{d(p, q)} - \sin \theta_{p,q} \right| < \epsilon.$$

where  $\theta_{p,q}$  denotes the infimum of  $\angle qpx$  when  $x$  runs over  $f^{-1}(f(p))$ .

We denote by  $\tau(\epsilon_1, \dots, \epsilon_k)$  a function depending on a priori constants and  $\epsilon_i$  satisfying  $\lim_{\epsilon_i \rightarrow 0} \tau(\epsilon_1, \dots, \epsilon_k) = 0$ .

**Theorem 1.2** (Fibration-Capping Theorem). *Given  $k$  and  $\mu > 0$  there exist positive numbers  $\delta = \delta_k$ ,  $\epsilon = \epsilon_k(\mu)$  and  $\nu = \nu_k(\mu)$  satisfying the following : Let  $X^k$  and  $M^n$  be as above, and let  $Y \subset R_\delta^D(X)$  be a closed domain such that  $\delta_D\text{-str.rad}(Y) \geq \mu$ . Suppose that  $X$  has only weak singularities, i.e.,  $M^n = R_{\delta_n}(M^n)$  for some small  $\delta_n > 0$ . If  $d_{GH}(M^n, X^k) < \epsilon$  for some  $\epsilon \leq \epsilon_n(\mu)$ , then there exists a closed domain  $N \subset M^n$  and a decomposition*

$$N = N_{\text{int}} \cup N_{\text{cap}}$$

*of  $N$  into two closed domains glued along their boundaries and a Lipschitz map  $f : N \rightarrow Y_\nu$  such that*

- (1)  $N_{\text{int}}$  is the closure of  $f^{-1}(\text{int}_0 Y_\nu)$ , and  $N_{\text{cap}} = f^{-1}(\partial_0 Y_\nu)$ ;
- (2) both the restrictions  $f_{\text{int}} := f|_{N_{\text{int}}} : N_{\text{int}} \rightarrow Y_\nu$  and  $f_{\text{cap}} := f|_{N_{\text{cap}}} : N_{\text{cap}} \rightarrow \partial_0 Y_\nu$  are
  - (a) locally trivial fibre bundles;
  - (b)  $\tau(\delta, \nu, \epsilon/\nu)$ -Lipschitz submersions.

*Remark 1.3.* (1) The fibres of both  $f_{\text{int}}$  and  $f_{\text{cap}}$  have almost nilpotent fundamental groups ([20]) and the first Betti numbers less than or equal to the dimensions of them ([46]).

- (2) If  $X$  has no boundary, then  $N = N_{\text{int}}$  and  $f = f_{\text{int}}$ . In this case, Theorem 1.2 is stated in [6], and will be called the fibration theorem as usual. See [47] and [43] in the case when  $M$  is a Riemannian manifold.

For instance, if  $X$  is a closed interval and if  $d_{GH}(M, X)$  is sufficiently small, then we have an obvious decomposition  $M = M_{\text{int}} \cup M_{\text{cap}}$ , where  $M_{\text{int}} \simeq F \times I$ ,  $F$  is a general fibre and  $M_{\text{cap}}$  consists of two components. If  $X$  is a Riemannian disk  $D^k$ , we have a decomposition  $M = M_{\text{int}} \cup M_{\text{cap}}$ , where  $M_{\text{int}} \simeq F \times D^k$  and  $M_{\text{cap}}$  is a fibre bundle over  $S^{k-1}$  whose fibre has boundary homeomorphic to  $F$ .

Applying the results of the present paper together with [20], [43], we have the following

**Corollary 1.4.** *Under the situation of Theorem 1.2, if the codimension  $m := \dim M - \dim X$  is less than or equal to three, the topology of the fibre  $F_{\text{cap}}$  of  $f_{\text{cap}}$  is described as follows:*

- (1) If  $m = 1$ ,  $F_{\text{cap}}$  is homeomorphic to  $D^2$ ;
- (2) If  $m = 2$ ,  $F_{\text{cap}}$  is homeomorphic to one of  $D^3$ ,  $P^2 \tilde{\times} I$ ,  $S^1 \times D^2$ , and  $K^2 \tilde{\times} I$ ;
- (3) If  $m = 3$ ,  $F_{\text{cap}}$  is homeomorphic to either a disk-bundle or a gluing of two disk-bundles having the same topology as  $U_i^4$  described in Theorem 0.9.

Note that when  $m \leq 3$ , the topology of the fibre of  $f_{\text{int}}$  is already determined in [46] and [20].

The proofs of Theorem 1.2 and Corollary 1.4 are deferred to Part 3, where we actually prove an equivariant version of Theorem 1.2 (see Theorem 18.4).

When no collapsing occurs, we have the following stability result due to [35].

**Theorem 1.5** (Stability Theorem [35]). *Let a sequence of compact  $n$ -dimensional Alexandrov spaces  $X_i$  with curvature  $\geq -1$  converge to a compact Alexandrov space  $X$  of dimension  $n$ . Then  $X_i$  is homeomorphic to  $X$  for sufficiently large  $i$ .*

In the proof of the theorem above, the notion of regular maps is crucial. Here we recall the most basic case of it.

Consider a map  $f = (f_1, \dots, f_m) : U \rightarrow \mathbb{R}^m$  on an open set  $U$  of an Alexandrov space  $X$  defined by the distance functions  $f_j(x) = d(A_j, x)$  from compact subsets  $A_j \subset X$ . The map  $f$  is said to be  $(c, \epsilon)$ -regular at  $p \in U$  if there is a point  $w \in X$  such that

- (1)  $\angle((A_j)'_p, (A_k)'_p) > \pi/2 - \epsilon$ ;
- (2)  $\angle(w'_p, (A_j)'_p) > \pi/2 + c$ ,

for every  $j \neq k$ . In [35], it was proved that if  $f = (f_1, \dots, f_m) : U \rightarrow \mathbb{R}^m$  is  $(c, \epsilon)$ -regular on  $U$  for some  $\epsilon > 0$  sufficiently small compared to  $c$ , then it is a topological submersion (see also [36]). We simply say that  $f$  is regular in this case. Together with [44], this implies that if  $f : U \rightarrow \mathbb{R}^m$  is regular and proper, then it is a locally trivial fibre bundle over its image.

Under the same assumption as Stability Theorem 1.5, suppose in addition that a regular map  $f : U \rightarrow \mathbb{R}^m$  is given as above on an open subset  $U \subset X$ . Using an approximation map  $X \rightarrow X_i$ , we can define a regular map  $f_i : U_i \rightarrow \mathbb{R}^m$  on an open subset  $U_i \subset X_i$  for any sufficiently large  $i$ .

The following is the respectful version of Stability Theorem 1.5.

**Theorem 1.6** ([35]). *Under the situation above, there exists a homeomorphism  $h_i : X \rightarrow X_i$  such that  $f_i \circ h_i = f$  holds on every compact set  $K \subset U$  and for sufficiently large  $i \geq i(f, K)$ .*

For any compact set  $A$  of an Alexandrov space  $X$ ,  $d_A = d(A, \cdot)$  denotes the distance function from  $A$ . In case  $X$  is a Riemannian manifold, we denote by  $\tilde{d}_A = \tilde{d}(A, \cdot)$  a smooth approximation of  $d_A$ .

## 2. PRELIMINARIES ON COMPLETE ALEXANDROV SPACES WITH NONNEGATIVE CURVATURE

Let  $C$  be a complete nonnegatively curved Alexandrov space with nonempty boundary. In [35], it was proved that the function  $d(\partial C, \cdot)$  is concave on  $C$ . We first show the rigidity for this function.

**Proposition 2.1.** *For a unit speed geodesic segment  $\gamma : [0, a] \rightarrow C$  joining  $p_0$  to  $p_1$ , suppose that  $d(\partial C, \gamma(t))$  is constant. Then for any minimal geodesic  $\gamma_0$  from  $p_0$  to  $\partial C$ , there is a minimal geodesic  $\gamma_1$  from  $p_1$  to  $\partial C$ , such that  $\{\gamma_0, \gamma, \gamma_1\}$  bounds a flat totally geodesic rectangle.*

*Proof.* This follows from a modification of Proposition 9.10 in [43]. For completeness, we give a proof below.

Let  $q_0 \in \partial C$  denote the end point of  $\gamma_0$ . We show that  $\Sigma_{q_0}(D(C))$  is the spherical suspension over  $\partial\Sigma_{q_0}(C)$ , where  $D(C)$  is the double of  $C$ . Let  $r_0 \in D(C)$  be another copy of  $p_0$ , and  $\xi_+$  and  $\xi_-$  the directions at  $q_0$  represented by geodesics  $q_0p_0$  and  $q_0r_0$  respectively. Obviously  $\angle(\xi_+, \xi_-) = \pi$ . It follows that  $\Sigma_{q_0}(D(C))$  is the spherical suspension over the hypersurface  $S := \{v \mid \angle(\xi_{\pm}, v) = \pi/2\}$ . Clearly  $\partial\Sigma_{q_0}(D(C))$  is a subset of  $\{v \mid \angle(\xi_{\pm}, v) \geq \pi/2\}$  which coincides with  $S$ . An obvious argument then implies that  $\partial\Sigma_{q_0}(D(C)) = S$ .

Let  $\sigma : [0, b] \rightarrow X$  be a unit speed minimal geodesic joining  $q_0$  to  $p_1$ . We consider the concave function  $f(u) = d(\sigma(u), \partial C)$ . Put  $\alpha = \angle(\dot{\sigma}(0), \partial\Sigma_{q_0}(C))$ , where  $\dot{\sigma}(0)$  denotes the direction at  $\sigma(0)$  represented by  $\sigma$ . From a standard argument,

$$f'(0) = \sin \alpha.$$

Consider the triangle  $\triangle q'_0p'_0p'_1$  on  $\mathbb{R}^2$  such that  $d(p'_0, q'_0) = t$ ,  $d(p'_0, p'_1) = a$  and  $\angle q'_0p'_0p'_1 = \pi/2$ , where  $t := d(p_0, q_0)$ . Set  $b' = d(q'_0, p'_1)$ ,  $\alpha' = \angle q'_0p'_1p'_0$ ,  $\theta' = \angle p'_0q'_0p'_1$ , and  $\theta = \angle p_0q_0p_1 = \pi/2 - \alpha$ . Since  $\angle q_0p_0p_1 = \pi/2$ , we have  $b' \geq b$ . It follows from the concavity of  $f$  that

$$f'(0) \geq \frac{t}{b} \geq \frac{t}{b'}.$$

Thus we obtain that

$$(2.1) \quad \alpha \geq \alpha' \quad \text{and} \quad \theta \leq \theta'.$$

Consider now a comparison triangle  $\tilde{\triangle} q_0p_0p_1$  in  $\mathbb{R}^2$  and put  $\tilde{\theta} = \angle p_0q_0p_1$ ,  $\tilde{\alpha} = \angle q_0p_1p_0$ . Since we may assume for our purpose that  $t > a$ , it follows from an obvious consideration with  $b > t$  that  $\alpha' \leq \tilde{\alpha} \leq \pi/2$ ,  $\theta' \leq \tilde{\theta}$  and hence

$$(2.2) \quad \theta' = \theta = \tilde{\theta}, \quad \alpha' = \tilde{\alpha} = \alpha, \quad b = b' \quad \text{and} \quad \angle q_0p_0p_1 = \pi/2.$$

It follows from the rigidity argument (cf. [42]) that  $\triangle q_0p_0p_1$  spans a totally geodesic flat triangle isometric to  $\tilde{\triangle} q_0p_0p_1$ . Furthermore,  $f'(0) = t/b$ . It follows from the concavity of  $f$  that  $f(u) = tu/b$  for all  $u$ . Let  $x_u$  and  $y_u$  be the points on  $\partial C$  and  $q_0p_1$  respectively such that  $f(u) = d(\sigma(u), x_u)$  and  $d(p_0, y_u) = ua/b$ . Then it follows together with the comparison argument that  $d(x_u, y_u) \leq d(x_u, \sigma(u)) + d(\sigma(u), y_u) \leq t$ . Thus  $\sigma$  lies on the minimal connections from the points of  $\gamma$  to  $\partial C$ .

By repeating the argument above for  $x_u, y_u, p_1$  in place of  $q_0, p_0, p_1$ , we conclude that the set of minimal connections  $x_u y_u$ ,  $0 \leq u \leq b$ , provides a totally geodesic flat rectangle.  $\square$

Let  $X$  be a complete noncompact Alexandrov space with nonnegative curvature. Consider the Busemann function associated with a reference point  $p \in X$  defined by

$$(2.3) \quad b(x) = \sup_{\gamma} b_{\gamma}(x),$$

where  $\gamma$  runs over all the geodesic rays emanating from  $p$  and

$$b_{\gamma}(x) = \lim_{t \rightarrow \infty} t - d(x, \gamma(t)).$$

Applying the Cheeger-Gromoll basic construction([11], [35]), we obtain a sequence of finitely many nonempty compact totally convex sets:

$$C(0) \supset C(1) \supset C(2) \supset \cdots \supset C(k),$$

where  $C(0)$  is the minimum set of  $b$ ,  $n > \dim C(0)$ ,  $\dim C(i) > \dim C(i+1)$  and  $C(k)$  has no boundary. Then a soul  $S$  of  $X$  is defined as  $S = C(k)$ . It was proved in [35] that  $X$  is homotopy equivalent to  $S$ .

The compact totally convex set  $C(0)$  depends on the reference point  $p$  and is denoted by  $C_p(0)$  for a moment. Now consider the integer

$$m_X := \inf_{p \in X} \dim C_p(0).$$

We denote by  $X(\infty)$  the ideal boundary of  $X$  equipped with the Tits distance.

**Lemma 2.2.** *If  $m_X = n - 1$ , then  $X^n(\infty)$  consists of at most two elements.*

*Proof.* Consider  $C(0)$  with an arbitrary reference point. By Proposition 2.1, we have a map  $f : C(0) \times \mathbb{R}$  satisfying:

- (1)  $f(x, 0) = x$  for any  $x \in C(0)$ ;
- (2) Both  $f(x \times \mathbb{R}_+)$  and  $f(x \times \mathbb{R}_-)$  represent geodesic rays from  $x$  for any  $x \in C(0)$ , where  $\mathbb{R}_+ = [0, \infty)$ ,  $\mathbb{R}_- = (-\infty, 0)$ ;
- (3) For any  $x, y \in C(0)$ ,  $f(x \times \mathbb{R}_{\pm})$ , a geodesic  $\gamma_{x,y}$  joining  $x$  and  $y$  and  $f(y \times \mathbb{R}_{\pm})$  span flat rectangles.

For any  $p \in \text{int } C(0)$ , consider  $C_p(0)$ . Clearly  $p \in C_p(0)$ . Let  $\gamma_{\pm}$  denote the geodesic rays  $f(p \times \mathbb{R}_{\pm})$ . Suppose that there is a geodesic ray  $\sigma$  starting from  $p$  different from  $\gamma_{\pm}$ , and consider the minimum set  $C$  of  $b_{\sigma}$  on the compact set  $C_p(0)$ . Since  $b_{\sigma}$  is locally nonconstant on  $C_p(0)$ ,  $\dim C < \dim C_p(0) = n - 1$ . Obviously  $C_q(0) \subset C$  for any  $q \in C$ , contradicting the assumption  $m_X = n - 1$ .  $\square$

**Example 2.3.** Let  $X$  be the double of the closed domain  $\{(x, y) \mid x, y \geq 0, y \geq x - 1\}$  on the  $(x, y)$ -plane. Note  $\dim X(\infty) = 1$ . For  $p = (1/2, 0)$ ,  $C_p(0)$  coincides with the segment  $[0, 1/2] \times 0$ , and for  $q = (0, 0)$ ,  $C_q(0) = \{q\}$ .

The above example shows that  $\dim C(0) = \dim X - 1$  does not imply  $\dim X(\infty) = 0$ .

**Proposition 2.4.**  $\dim X(\infty) \leq \operatorname{codim} S - 1$ .

Independently Shioya has obtained the above proposition in a similar way.

**Lemma 2.5.** *Let  $\Sigma$  be an compact Alexandrov space with curvature  $\geq 1$ , and  $\Sigma_0 \subset \Sigma$  be a closed locally convex set of positive dimension and without boundary as an Alexandrov space. Let  $\xi \in \Sigma$  be such that  $d(\xi, \Sigma_0) \geq \pi/2$ . Then  $d(\xi, v) = \pi/2$  for every  $v \in \Sigma_0$ .*

*Proof.* Let  $v_0 \in \Sigma_0$  be a foot from  $\xi$  to  $\Sigma_0$ . We proceed by induction on  $\dim \Sigma_0$ . If  $\dim \Sigma_0 = 1$ , then  $\Sigma_0$  is isometric to a circle. Since  $\angle \xi v_0 v = \pi/2$ , the lemma follows from the Alexandrov convexity.

Suppose the lemma is true for  $\dim \Sigma_0 - 1$ . Put  $\Sigma' = \Sigma_{v_0}(\Sigma)$ ,  $\Sigma'_0 = \Sigma_{v_0}(\Sigma_0)$ . Let  $\xi' = \xi'_{v_0} \in \Sigma'$ ,  $v' = v'_{v_0} \in \Sigma'_0$  for every  $v \in \Sigma_0$ . It is easy to check that  $\Sigma'_0$  is a closed locally convex subset of  $\Sigma'$ . Applying the induction hypothesis, we see that  $\angle(\xi', v') = \pi/2$  for every  $v \in \Sigma_0$ . The Alexandrov convexity then implies the conclusion.  $\square$

*Proof of Proposition 2.4.* It suffices to show that for any regular point  $p$  of  $S$  and any geodesic ray  $\gamma$  from  $p$ ,  $\dot{\gamma}(0)$  is perpendicular to  $S$ . Let  $x \in S$  be a foot from  $\xi = \gamma(\infty) \in X(\infty)$  to  $S$ , and  $\sigma$  a geodesic ray from  $x$  asymptotic to  $\gamma$ . Namely we have sequences  $x_n \in S$ ,  $y_n \in X$  satisfying

- (1)  $x_n \rightarrow x$ ,  $y_n \rightarrow \xi$ ,  $x_n y_n \rightarrow \sigma$ ;
- (2)  $d(y_n, x_n) = d(y_n, S)$ .

Note that  $\Sigma_x(S)$  is a closed locally convex set of  $\Sigma_x(X)$ . By the previous lemma,  $\lim \angle q x_n y_n = \pi/2$  for any  $q \in S$ , and hence  $\angle q x \xi = \pi/2$ . The Alexandrov convexity then implies that

$$(2.4) \quad \tilde{\angle} x q y_n > \pi/2 + o_n$$

with  $\lim_{n \rightarrow \infty} o_n = 0$ . Thus we have  $\lim_{n \rightarrow \infty} |d(y_n, q) - d(y_n, x)| = 0$ , and hence  $b_\sigma(x) = b_\sigma(q)$ , where  $b_\sigma$  denotes the Busemann function associated with the ray  $\sigma$ . It follows that  $b_\gamma(x) = b_\gamma(q)$  for any  $q \in S$ . Thus  $\gamma$  must be perpendicular to  $S$  at  $p$ .  $\square$

We denote by  $B(S, \epsilon)$  the closed  $\epsilon$ -metric ball around  $S$ .

**Theorem 2.6** (Generalized Soul Theorem). *Let  $X$  be a 4-dimensional complete open (i.e., noncompact and boundaryless) Alexandrov space with nonnegative curvature. Suppose in addition that  $X$  is topologically regular. Then there exists a positive number  $\epsilon$  such that*

- (1)  $X$  is homeomorphic to  $\operatorname{int} B(S, \epsilon)$ ;
- (2)  $B(S, \epsilon)$  is homeomorphic to a disk-bundle over  $S$ , called the normal bundle of  $S$ .



The proof of Theorem 2.6 is given in Part 2.

**Corollary 2.7.** *Suppose that a sequence of 4-dimensional pointed complete Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  converges to a pointed complete noncompact 4-dimensional Alexandrov space  $(Y^4, y_0)$  with nonnegative curvature. Then for a sufficiently large  $R > 0$ ,  $B(p_i, R)$  is homeomorphic to the normal closed disk-bundle over the soul of  $Y^4$ .*

*Proof.* Note first that  $Y^4$  is topologically regular ([29]). By Theorem 1.6,  $B(p_i, R)$  is homeomorphic to  $B(y_0, R)$  any sufficiently large  $R$  because of the nonnegativity of the curvature of  $Y^4$ . By Theorem 2.6,  $B(y_0, R)$  is homeomorphic to the normal closed disk-bundle over the soul of  $Y^4$   $\square$

### 3. $S^1$ -ACTIONS ON ORIENTED 4-MANIFOLDS

In this section, we present some results on  $S^1$ -actions on oriented 4-manifolds which will be used in the subsequent sections.

First we fix the notation and terminology about local  $S^1$ -actions. Suppose that an open covering  $\{U_\alpha\}$  of a manifold  $N$  and an  $S^1$ -action  $\psi_\alpha$  on  $U_\alpha$  are given in such a way that both the actions  $\psi_\alpha$  and  $\psi_\beta$  coincide up to orientation on the intersection  $U_\alpha \cap U_\beta$ . We say that  $\{(U_\alpha, \psi_\alpha)\}$  defines a local  $S^1$ -action, denoted  $\psi$ , on  $N$ . The local  $S^1$ -action  $\psi$  is *locally smooth* if each  $x \in N$  has a slice  $V_x$  which is a disk invariant under the action of the isotropy group  $S_x^1$  at  $x$  and if this action is equivariant to an orthogonal action.

Let  $N^* = N/\psi$  be the orbit space and  $\pi : N \rightarrow N^*$  the orbit map. An orbit is called *exceptional* if the isotropy group at a point on the orbit is non-trivial but finite. Let  $F = F(\psi)$  and  $E = E(\psi)$  denote the fixed point set and the union of the exceptional orbits respectively. In the present paper, an orbit in  $S = S(\psi) := F \cup E$  is called a singular orbit. The images  $F^* := \pi(F(\psi))$ ,  $E^* := \pi(E(\psi))$  and  $S^* := \pi(S(\psi))$  are called the fixed point locus, the exceptional locus and the singular locus respectively.

**Lemma 3.1.** (1) *If  $N$  is simply connected, then so is  $N^*$ ;*  
(2) *If both  $N$  and  $N^* - S^*$  are orientable, then  $\psi$  is actually an  $S^1$ -action.*

*Proof.* (1). Let  $\gamma^*$  be any loop at a point  $p^* \in N^* - S^*$ . Since  $S^* \cap \text{int } N^*$  is of codimension  $\geq 2$ , one can slightly perturb  $\gamma^*$  to a loop  $\sigma^*$  in  $N^* - S^*$ . Let  $\sigma$  be a lift of  $\sigma^*$  to  $N - S$ . From the assumption,  $\sigma$  is homotopic to a path in  $\pi^{-1}(p^*)$  keeping endpoints fixed. Thus  $\sigma^*$  and hence  $\gamma^*$  is null-homotopic.

(2). For each  $x \in N - S$ , we have the orientation on the slice  $V_x$  at  $x$  induced from the orientation of  $N^* - S^*$ . Then the orientation of  $N$  induces the orientation of the orbit  $S^1x$  in such a way that the orientation of  $V_x$  followed by the orientation of  $S^1x$  coincides with the

original orientation of  $N$ . Since  $E^*$  is of codimension  $\geq 2$ , those orientations of the orbits in  $N - S$  extend to the orientations of the orbits in  $N - F$ .  $\square$

**Definition 3.2.** Let  $D^4(1) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ , and let  $a, b$  be relatively prime integers. The  $S^1$ -action  $\psi_{a,b}$  on  $D^4(1)$  defined by

$$z \cdot (z_1, z_2) = (z^a z_1, z^b z_2),$$

is called the *canonical  $S^1$ -action of type  $(a, b)$* . For any  $0 < t \leq 1$ , the restriction of  $\psi_{a,b}$  to  $S^3(t) = \{(z_1, z_2) \in D^4(1) \mid |z_1|^2 + |z_2|^2 = t^2\}$  gives a Seifert bundle  $S^3(t) \rightarrow S^3(t)/\psi_{a,b} \simeq S^2$ .

Note that the restriction  $\psi_{1,1}|_{S^3(t)}$  is the Hopf-fibration. Note also that the orbit space  $D^4(1)/\psi_{a,b}$  with the standard quotient metric is an Alexandrov space with nonnegative curvature whose space of directions  $\Sigma_o$  at the origin  $o \in D^4(1)/\psi_{a,b}$  has at most two singular points, say  $\xi_1$  and  $\xi_2$ , where

$$\{|a|, |b|\} = \{2\pi/L(\Sigma_{\xi_1}(\Sigma_o)), 2\pi/L(\Sigma_{\xi_2}(\Sigma_o))\}.$$

We describe the equivariant classification of  $S^1$ -actions on oriented 4-manifolds  $M$  slightly extending the treatment in [16]. In [16],  $M$  is assumed to be simply connected and to have no boundary. In our later use however, we need the more general case when  $M$  has nonempty boundary and only the orbit space  $M^*$  is assumed to be simply connected. For convenience, we describe the equivariant classification in our general setting.

Let  $M$  be an oriented 4-manifold (possibly with boundary) with a locally smooth  $S^1$ -action. We assume that there are no fixed points on  $\partial M$ . Let  $\pi : M \rightarrow M^*$  be the orbit map. The proof of the following Proposition is similar to Proposition (3.1) in [16], and hence omitted.

**Proposition 3.3.** *Under the situation above,  $M^*$  is a 3-manifold for which the general properties of the singular locus are described as follows:*

- (1)  $\partial M^* - \pi(\partial M) \subset F^*$ ;
- (2)  $F^* - \partial M^*$  is discrete;
- (3) *The closure  $\bar{E}^*$  of  $E^*$  is a disjoint union of arcs and simply closed curves in  $M^* - (\partial M^* - \pi(\partial M))$ . Each connected component of  $E^*$  is one of the following;*
  - (a) *A simple closed curve in  $\text{int } M^*$ ;*
  - (b) *An arc whose closure joins two points of  $(F^* \cap \text{int } M^*) \cup \pi(\partial M)$ .*

Note that the orbit type is constant on each component of  $E^*$ . From now on we assume that  $M^*$  is simply connected.

**Lemma 3.4.** *If  $M^*$  is simply connected, then each component of  $F^* \cap \partial M^*$  is homeomorphic to either  $S^2$  or  $D^2$ .*

*Proof.* Note that each component of  $\partial M^*$  is a sphere. On the double  $D(M)$  we have an  $S^1$ -action induced from the original  $S^1$ -action. By Van Kampen's theorem, the orbit space  $D(M)^*$  is simply connected. Suppose that there is a component  $F_0^*$  of  $F^* \cap \partial M^*$  which is neither a disk nor a sphere. Then it turns out the double  $D(F_0^*)$  is a component of  $\partial(D(M)^*)$  and a surface with genus  $\geq 1$ , a contradiction to the simple connectivity of  $D(M)^*$ .  $\square$

The orientation of  $M$  induces an orientation of  $M^*$  in such a way that the orientation of  $M^*$  followed by the natural orientation of the orbit coincides with the original orientation of  $M$ . For a given oriented submanifold  $X^*$  of  $M^*$  we also use this convention to orient  $\pi^{-1}(X^*)$ .

We assign the following orbit data to  $M^*$ :

- (1) For a given component  $F_i^*$  of  $F^* \cap \partial M^*$  homeomorphic to  $S^2$ , take a collar neighborhood  $F_i^* \times [0, 1]$  of  $F_i^*$ , and orient  $F_i^* \times \{1\}$  by the outward normal. We assign to  $F_i^*$  the Euler number of the  $S^1$ -bundle  $\pi^{-1}(F_i^* \times \{1\}) \rightarrow F_i^* \times \{1\}$ ;
- (2) For each  $x^* \in F^* \cap \text{int } M^* - \bar{E}^*$ , take a small disk neighborhood  $B^*$  of  $x^*$ . We assign to  $x^*$  the Euler number  $+1$  or  $-1$  of the  $S^1$ -bundle  $\pi^{-1}(\partial B^*) \simeq S^3 \rightarrow \partial B^* \simeq S^2$ ;
- (3) Let  $L^*$  be a simple closed curve in  $E^* \cup (F^* \cap \text{int } M^*)$  and fix an orientation of  $L^*$ . For each component  $J^*$  of  $L^* - F^*$ , let  $y^*$  be the end point of  $J^*$ , choose a sufficiently small disk domain  $B^*$  of  $y^*$  and orient  $\partial B^*$  by a normal with direction of  $J^*$ . We assign to  $J^*$  the Seifert invariant  $(\alpha, \beta)$  of the orbit in  $\pi^{-1}(\partial B^*)$  with image  $J^* \cap \partial B^*$ . The weights assigned to  $L^*$  consists of the orientation and a finite system of the Seifert invariants, abbreviated by  $\{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$ . If the orientation of  $L^*$  is reversed, the finite system changes to  $\{(\alpha_n, \alpha_n - \beta_n), \dots, (\alpha_1, \alpha_1 - \beta_1)\}$ . We regards these two weights to be equivalent;
- (4) Let  $A^*$  be an arc component of  $E^* \cup (F^* \cap \text{int } M^*)$ . Choosing an orientation of  $A^*$ , we define a finite system of Seifert invariants as in (3). Let  $y^*$  be either the initial point or the end point of  $A^*$ , and choose a sufficiently small disk domain  $B^*$  of  $y^*$  and orient  $\partial B^*$  as in (3). We assign to  $y^*$  the obstruction number  $b$  for the Seifert bundle  $\pi^{-1}(\partial B^*) \rightarrow \partial B^*$  with exactly one singular orbit. The weights assigned to  $A^*$  consists of the orientation, a finite system of the Seifert invariants and the two obstruction numbers at the end points, abbreviated by  $\{b'; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n); b''\}$ . If the orientation of  $A^*$  is reversed, the weights change to  $\{-1 - b'''; (\alpha_n, \alpha_n - \beta_n), \dots, (\alpha_1, \alpha_1 - \beta_1); -1 - b'\}$ . We regards these two weights to be equivalent.

$M^*$  together with the above collection of weights is called a *weighted orbit space*.

**Lemma 3.5** ([16]). *The weights defined above have some restriction stated below:*

- (1) *If  $(\alpha_i, \beta_i)$  and  $(\alpha_{i+1}, \beta_{i+1})$  denotes the Seifert invariants assigned to adjacent arcs in some weighted arc or circle, then*

$$\det \begin{pmatrix} \alpha_i & \beta_i \\ \alpha_{i+1} & \beta_{i+1} \end{pmatrix} = \pm 1.$$

- (2) *If  $\{b'; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n); b''\}$  denotes the weights of a weighted arc, then*

$$b' \alpha_1 + \beta_1 = \pm 1, \quad b'' \alpha_n + \beta_n = \pm 1.$$

**Proposition 3.6** ([16]). *Let  $M_1$  and  $M_2$  be compact oriented 4-manifolds (possibly with boundary) with  $S^1$ -action such that their weighted orbit spaces are isomorphic. Then  $M_1$  is orientation-preservingly equivariantly homeomorphic to  $M_2$ .*

*Proof.* This can be proved by the same methods as [16] by using Propositions (3.6), (3.9), Lemma (6.1) and Theorem (6.2) in [16] together with Proposition (9.1) in [17].  $\square$

Next we discuss  $S^1$ -actions on  $D^2$ -bundles over  $S^2$  (see [32] for details), which will be used in the subsequent sections to construct local structures of collapsing in several cases.

Let  $S^2 = B_1 \cup B_2$  be the union of its upper and lower hemispheres, and  $D_j^2 = D^2(1)$  the unit disks,  $j = 1, 2$ . Put polar coordinates on  $B_j \times D_j^2$ . For an arbitrary integer  $\omega$ , we consider the  $D^2$ -bundle over  $S^2$  defined as

$$S^2 \tilde{\times}_\omega D^2 := B_1 \times D_1^2 \bigcup_{f_\omega} B_2 \times D_2^2,$$

where  $f_\omega : \partial B_1 \times D_1^2 \rightarrow \partial B_2 \times D_2^2$  is the gluing homeomorphism defined by

$$f_\omega(e^{i\varphi}, se^{i\phi}) = (e^{-i\varphi}, se^{i(-\omega\varphi+\phi)}).$$

This bundle has Euler number  $\omega$ , which coincides with the self-intersection number of the zero section. Note that the boundary of  $S^2 \tilde{\times}_\omega D^2$  is homeomorphic to the lens space  $L(|\omega|, 1)$ . Conversely any  $D^2$ -bundle over  $S^2$  with the self-intersection number  $\omega$  of its zero section is homeomorphic to  $S^2 \tilde{\times}_\omega D^2$ .

For relatively prime integers  $a_j$  and  $b_j$ , define an  $S^1$ -action  $\psi_j$  on  $B_j \times D_j^2$  by

$$e^{i\theta}(re^{i\varphi}, se^{i\phi}) = (re^{i(\varphi+a_j\theta)}, se^{i(\phi+b_j\theta)}).$$

Those actions  $\psi_j$  define an  $S^1$ -action, denoted  $\hat{\psi}(a_1, b_1)$ , on the bundle  $S^2 \tilde{\times}_\omega D^2$  if and only if  $a_2 = -a_1$  and  $b_2 = -\omega a_1 + b_1$ .

We show that some  $S^1$ -actions with simple orbit spaces essentially come from  $S^1$ -actions on  $S^2 \tilde{\times}_\omega D^2$  with suitable  $\omega$  :

**Proposition 3.7.** *Let  $M$  be a compact oriented 4-manifold with boundary and with a locally smooth  $S^1$ -action  $\psi_0$  whose orbit space  $M^*$  is homeomorphic to  $D^3$ . Suppose that*

- (1) *the closure of any component of the exceptional locus  $E^*$  of  $\psi_0$  meets  $F^* \cap \text{int } M^*$ ;*
- (2)  *$F^*$  consists of  $m$  points of  $\text{int } M^*$  and  $n$  pieces of disjoint disks on  $\partial M^*$  with  $m + n = 2$ ,  $m, n \geq 0$ .*

*Then there is an  $S^1$ -action  $\psi$  on  $S^2 \tilde{\times}_\omega D^2$  as described above which is equivariantly homeomorphic to  $(M, \psi_0)$ , where  $\omega$  is determined by the weighted orbit invariants of  $\psi_0$  as follows:*

- (a) *If  $m = 2$  and  $E^*$  is empty, then  $\omega \in \{1, 2\}$ ;*
- (b) *If  $m = 2$  and  $E^*$  is an oriented arc with Seifert invariants  $(\alpha'', \beta'')$  joining a point of  $F^*$  to a point of  $\partial M^*$ , then  $|\omega| = \alpha'' \pm 1$ ;*
- (c) *Suppose that  $m = 2$  and  $E^*$  consists of two oriented arcs  $J'$  and  $J''$  with Seifert invariants  $(\alpha', \beta')$  and  $(\alpha'', \beta'')$  respectively, where  $J'$  joins a point of  $\partial M^*$  to a point of  $F^*$  and  $J''$  joins the other point of  $F^*$  to a point of  $\partial M^*$ . Then  $|\omega| = |\alpha' \pm \alpha''|$ ;*
- (d) *If  $m = 2$  and  $E^*$  is an arc joining the two points of  $F^*$ , then  $\omega = 0$ ;*
- (e) *If  $m = 2$  and  $E^*$  consists of an oriented arc  $J$  joining the two points of  $F^*$  with Seifert invariants  $(\alpha, \beta)$ , and an oriented arc from the end point of  $J$  to a point of  $\partial M^*$  with Seifert invariants  $(\alpha'', \beta'')$ , then  $|\omega| = (\alpha'' \pm 1)/\alpha$ ;*
- (f) *Suppose that  $m = 2$  and  $E^*$  consists of three oriented arcs  $J'$ ,  $J$  and  $J''$  with Seifert invariants  $(\alpha', \beta')$ ,  $(\alpha, \beta)$  and  $(\alpha'', \beta'')$  respectively, where  $J'$  joins a point of  $\partial M^*$  to a point of  $F^*$ ,  $J$  joins the end point of  $J'$  to the other point of  $F^*$  and  $J''$  joins the end point of  $J$  to a point of  $\partial M^*$ . Then  $|\omega| = |\alpha' \pm \alpha''|/\alpha$ ;*
- (g) *If  $m = n = 1$  and  $E^*$  is empty, then  $|\omega| = 1$ ;*
- (h) *If  $m = n = 1$  and  $E^*$  is an arc from a point of  $\partial M^*$  to  $F^* \cap \text{int } M^*$  with Seifert invariants  $(\alpha', \beta')$ , then  $|\omega| = \alpha'$ ;*
- (i) *If  $n = 2$ , then  $\omega = 0$ .*

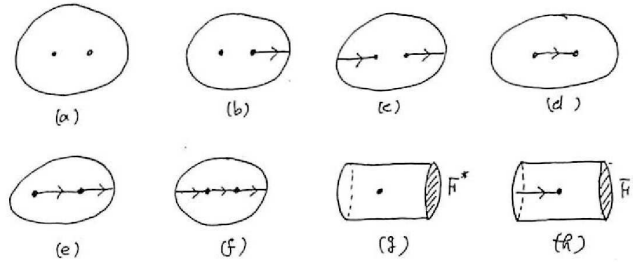


FIGURE 1.

*Proof.* This follows from Proposition 3.7 and the examples of  $S^1$ -actions on the  $D^2$ -bundles over  $S^2$  given in Section 4 of [16].  $\square$

*Remark 3.8.* In Proposition 3.7, if  $m+n=1$ , then  $M$  is homeomorphic to  $D^4$  and  $(M, \psi_0)$  is equivariantly homeomorphic to either  $(D^4, \psi_{a,b})$  for some relatively prime integers  $a$  and  $b$  ( $m=1$ ) or  $(D^2 \times D^2, 1 \times \text{rotation})$  ( $n=1$ ).

## Part 1. Reproducing collapsed 4-manifolds

In this Part 1, we use some results from Parts 2 and 3, for instance Theorem 2.6 and Theorem 18.4.

### 4. RESCALING ARGUMENT

We denote by  $\tau(a_1, \dots, a_k | \epsilon)$  a function depending on a priori constants,  $a_1, \dots, a_k$  and  $\epsilon$  satisfying  $\lim_{\epsilon \rightarrow 0} \tau(a_1, \dots, a_k | \epsilon) = 0$  for each fixed  $a_1, \dots, a_k$ .

For a closed set  $S$  in a metric space, we denote by  $A(S; r_1, r_2)$  the annulus  $B(S, r_2) - \text{int } B(S, r_1)$ .

Let a sequence of pointed complete  $n$ -dimensional Riemannian manifolds  $(M_i^n, p_i)$  with  $K \geq -1$  converge to a pointed  $k$ -dimensional Alexandrov space  $(X^k, p)$  with respect to the pointed Gromov-Hausdorff convergence, where  $k \leq n-1$ . In this section, we establish a basic result to study the topology of the metric ball  $B(p_i, r)$  for any sufficiently large  $i$  compared to a fixed small  $r > 0$ . If  $p$  is a regular point of  $X^k$ , then  $B(p_i, r)$  fibers over  $B(p, r)$  with almost nonnegatively curved fibre ([47]). Hence from now on, we assume that  $p$  is a singular point.

The purpose of this section is to prove the following result.

**Theorem 4.1.** *Suppose  $n=4$  and that  $B(p_i, r)$  is not homeomorphic to  $D^4$  for each  $r > 0$  and for any sufficiently large  $i$ . Then there exist an  $r = r_p > 0$  and sequences  $\delta_i \rightarrow 0$  and  $\hat{p}_i \in B(p_i, r)$  such that*

- (1)  $\hat{p}_i \rightarrow p$  under the convergence  $B(p_i, r) \rightarrow B(p, r)$ ;
- (2)  $B(p_i, r)$  is homeomorphic to  $B(\hat{p}_i, R\delta_i)$  for every  $R \geq 1$  and large  $i$  compared to  $R$ ;
- (3) for any limit  $(Y, y_0)$  of  $(\frac{1}{\delta_i} M_i^4, \hat{p}_i)$ , we have  $\dim Y \geq k+1$ .

*Remark 4.2.* (1) The homeomorphism in Theorem 4.1 (2) is given by the flow curves of a gradient-like vector field of the distance function  $d_{\hat{p}_i} = d(\hat{p}_i, \cdot)$ ;

- (2) The rescaling constant  $\delta_i$  can be thought of as the size of the (singular) fibre which is not visible yet.
- (3) The above theorem has been proved in [43] under the hypothesis that  $k=2$  and  $\text{diam}(\Sigma_p) < \pi$ ;
- (4) A generalization of Theorem 4.1 actually holds in the general dimension. This will appear in a forthcoming paper.

Let  $(Y, y_0)$  be a complete noncompact Alexandrov space with non-negative curvature given in Theorem 4.1, and  $Y(\infty)$  denote the ideal boundary of  $Y$  with the Tits metric.

The following lemma has been proved in Lemma 3.7 of [43].

**Lemma 4.3.** *There is an expanding map  $\Sigma_p \rightarrow Y(\infty)$ . In particular,  $\dim Y(\infty) \geq \dim \Sigma_p$ .*

Actually we have the following lemma in more general setting. The proof is similar to Lemma 3.7 of [43], and hence omitted.

**Lemma 4.4.** *Suppose that a sequence  $(X_i^n, x_i)$  of  $n$ -dimensional complete noncompact pointed Alexandrov spaces with curvature  $\geq -\epsilon_i$ ,  $\lim_{i \rightarrow \infty} \epsilon_i = 0$ , converges to a pointed space  $(Z, z_0)$ . For  $\mu_i > 0$  converging to 0, let  $(W, w_0)$  be any limit of  $(\frac{1}{\mu_i} X_i^n, x_i)$ . Then we have an expanding map  $\Sigma_{z_0} \rightarrow W(\infty)$ .*

*In particular,  $\dim W(\infty) \geq \dim Z - 1$ .*

We go back to the situation of Theorem 4.1. Let  $S$  be a soul of  $Y$ .

**Proposition 4.5.**  $\dim S \leq \dim Y - \dim X$ .

This immediately follows from Proposition 2.4 and Lemma 4.3.

As a preliminary to the proof of Theorem 4.1, we recall an argument in Section 3 of [43].

For a compact set  $A \subset X^k$  and  $\epsilon > 0$ , let  $\beta_A(\epsilon)$  denote the maximal number of points in  $A$  having distance  $\geq \epsilon$ .

Take a  $\mu_i$ -approximation  $\phi_i : B(p, 1/\mu_i) \rightarrow B(p_i, 1/\mu_i)$  with  $\phi_i(p) = p_i$ , where  $\mu_i \rightarrow 0$  as  $i \rightarrow \infty$ . For any  $\epsilon > 0$ , we take an  $\epsilon$ -net  $\{\xi^j\}_{j=1, \dots, \beta_{\Sigma_p}(\epsilon)}$  of  $\Sigma_p$ . For a small enough  $r > 0$  compared to  $p$  and  $\epsilon$ , take  $x^j \in \partial B(p, r)$ ,  $j = 1, \dots, \beta_{\Sigma_p}(\epsilon)$ , such that the direction  $\eta^j$  at  $p$  of a minimal segment from  $p$  to  $x^j$  satisfies  $\angle(\xi^j, \eta^j) < \tau(r) < \epsilon^2$ . Set  $x_i^j = \phi_i(x^j)$  and

$$(4.1) \quad f_i = f_{\epsilon, r, i} = \frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j \frac{1}{r} d(x_i^j, \cdot) : M_i^n \rightarrow \mathbb{R}.$$

Consider the measure  $m_\epsilon = \frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j \delta_{\xi^j}$ , where  $\delta_x$  is the Dirac  $\delta$ -measure. We have a sequence  $\epsilon_\ell \rightarrow 0$  such that the measure  $m_{\epsilon_\ell}$  converges to some Borel measure  $m_p$  on  $\Sigma_p$  as  $\ell \rightarrow \infty$  in the weak\* topology. Note that

- (1)  $m_p$  coincides with the normalized Hausdorff measure over  $\Sigma_p$  if  $k \leq 2$ ;
- (2)  $m_p$  is regular in the sense that  $m_p(U) > 0$  for any non-empty open subset  $U$  of  $\Sigma_p$ .

The latter property comes from the Bishop-Gromov volume comparison theorem (cf.[47]) for  $\Sigma_p$ .

We make the identification  $\Sigma_p = \Sigma_p \times \{1\} \subset K_p$ . Let  $\psi_r : B(o_p, 1/\nu_r) \rightarrow B(p, \nu_r; \frac{1}{r}X^k)$  be a  $\nu_r$ -approximation such that  $\psi_r(o_p) = p$  and  $\psi_r(\xi^j) = x^j$ , where  $\lim_{r \rightarrow 0} \nu_r = 0$ , and let

$$\bar{f} = \int_{\xi \in \Sigma_p} d(\xi, \cdot) dm_p : K_p \rightarrow \mathbb{R}.$$

We then have

$$(4.2) \quad |f_i \circ \phi_i \circ \psi_r - \bar{f}| < \tau(\ell, r|1/i) + \tau(\ell|r) + \tau(1/\ell),$$

on any fixed compact set of  $K_p$ , simply denoted as

$$\lim_{\ell \rightarrow \infty} \lim_{r \rightarrow 0} \lim_{i \rightarrow \infty} \bar{f}_i \circ \phi_i \circ \psi_r = \bar{f}.$$

**Lemma 4.6** ([43], Lemmas 3.4, 3.5). *If  $\bar{f}$  takes a strictly local maximum at the vertex  $o_p \in K_p$ , then we can find  $\delta_i \rightarrow 0$  and  $\hat{p}_i \in B(p_i, r)$  satisfying the conclusion of Theorem 4.1.*

We give a sketch of the essential idea of the proof of Lemma 4.6.

From the assumption, one can take as the point  $\hat{p}_i$  as a local maximum point of  $f_i$  converging to  $o_p$ , and as  $\delta_i$  to be the maximum distance between  $\hat{p}_i$  and the critical point set ([26], [22]) of  $d_{\hat{p}_i}$  within  $B(p, r)$ . Let  $\hat{q}_i$  be a critical point of  $d_{\hat{p}_i}$  within  $B(p, r)$  realizing  $\delta_i$ , and let  $z_0 \in Y$  be the limit of  $\hat{q}_i$  under the convergence  $(\frac{1}{\delta_i}M_i, \hat{p}_i) \rightarrow (Y, y_0)$ . Let  $x_\infty^j \in Y(\infty)$  denote the point at infinity defined by the limit ray, say  $\gamma_j$  from  $y_0$  of the geodesic  $\hat{p}_i x_i^j$  under the convergence  $(\frac{1}{\delta_i}M_i, \hat{p}_i) \rightarrow (Y, y_0)$ . For fixed  $\epsilon = \epsilon_\ell$  and  $r$ ,  $f_i$  after some normalization converges to the function  $g = \frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j b_{x_\infty^j}$ , where  $b_{x_\infty^j}$  can be thought of as a generalized Busemann function associated with  $\gamma_j$ . Let  $v \in \Sigma_{y_0}$  be a direction of a minimal geodesic from  $y_0$  to  $z_0$ , and let  $\theta_j$  denote the angle between  $v$  and  $\gamma_j$ . Since  $z_0$  is a critical point of  $d_{y_0}$  and since  $Y$  has nonnegative curvature, we obtain that  $\theta_j \geq \pi/2$ . On the other hand, since  $y_0$  is a maximum point of  $g$ , we have

$$0 \geq g'_{y_0}(v) = -\frac{1}{\beta_{\Sigma_p}(\epsilon)} \sum_j \cos \theta_j \geq 0.$$

It follows that  $\theta_j = \pi/2$ . Therefore  $\{w_j\}_{1 \leq j \leq \beta_{\Sigma_p}(\epsilon)}$ , which have pairwise distance  $\geq \epsilon$ , must be contained in a metric sphere  $\partial B(v, \pi/2; \Sigma_{y_0})$ . Since this holds for any sufficiently small  $\epsilon$  with  $\beta_{\Sigma_p}(\epsilon) \sim \text{const} \epsilon^{\dim X - 1}$ , we can conclude  $\dim Y \geq k + 1$ .

**Lemma 4.7.** *Suppose one of the following two cases.*

- (1)  $k = 2$  and  $\text{diam}(\Sigma_p) < \pi$ ;
- (2)  $\text{diam}(\Sigma_p) \leq \pi/2$ .

*Then the function  $\bar{f}$  takes a strictly local maximum at  $o_p \in K_p$ . In particular, Theorem 4.1 holds in those cases.*



*Proof.* The case (1) is proved in [43], and Suppose the case (2) and that there exists a sequence  $x_i \rightarrow o_p$  satisfying  $\bar{f}(x_i) \geq \bar{f}(o_p)$  and take  $\xi_i \in \Sigma_p$  with  $1 = d(o_p, \xi_i) = d(o_p, x_i) + d(x_i, \xi_i)$ . Putting  $r_i = d(o_p, x_i)$ , we have  $d_\xi(x_i) \leq 1 - r_i/2$  for every  $\xi \in \Sigma_p$  with  $d(\xi, \xi_i) \leq \delta$ , where  $\delta$  is a small positive constant independent of  $i$ . Now by the assumption,  $d_\eta(x_i) \leq 1 + O(r_i^2)$  for all  $\eta \in \Sigma_p$ . It follows that

$$\begin{aligned} \bar{f}(x_i) &= \int_{B(\xi, \delta; \Sigma_p)} d(\xi, x_i) dm_p + \int_{B(\xi, \delta; \Sigma_p)^c} d(\xi, x_i) dm_p \\ &\leq (1 - r_i/2)m_p(B(\xi, \delta; \Sigma_p)) + (1 + O(r_i^2))m_p(B(\xi, \delta; \Sigma_p)^c) \\ &\leq (1 + O(r_i^2))m_p(\Sigma_p) - \frac{r_i}{2}m_p(B(\xi, \delta; \Sigma_p)) \\ &< m_p(\Sigma_p) = \bar{f}(o_p), \end{aligned}$$

for large  $i$ . This is a contradiction.  $\square$

**Lemma 4.8.** *Suppose  $k \leq 3$ . If  $\text{diam}(\Sigma_p) = \pi$ , then Theorem 4.1 holds.*

*Proof.* We first assume that  $\dim X = 3$  and  $p$  is not a boundary point of  $X^3$ . Note that  $K_p = K(\Sigma) \times \mathbb{R}$ , where  $\Sigma$  is a circle of length  $< 2\pi$ . To apply the previous argument to  $K(\Sigma)$ , let us consider the function

$$\tilde{f} = \int_{\xi \in \Sigma} d(\xi, \cdot) d\mathcal{H} : K(\Sigma) \rightarrow \mathbb{R},$$

where  $d\mathcal{H}$  denotes the normalized Hausdorff measure of  $\Sigma$ . By Lemma 4.7,  $\tilde{f}$  takes a strictly local maximum at  $o_p$ . Take  $\tau_r > 0$  with  $\lim_{r \rightarrow 0} \frac{\tau_r}{\nu_r} = \infty$ , and for a point  $v_r$  on the line  $o_p \times \mathbb{R} \subset K(\Sigma) \times \mathbb{R}$  with  $d(v_r, o_p) = 1/\tau_r$ , let  $w_r := \psi_r(v_r)$  and  $w_r^i := \phi_i(w_r)$ . Let  $S_r^i$  be the three-dimensional submanifold defined as

$$S_r^i := \{d_{w_r^i} = d_{w_r^i}(p_i)\} \cap \{d_{p_i} \leq r\}.$$

We construct a function  $\tilde{f}_i$  on  $M_i^4$  by using an  $\epsilon_\ell$ -net  $\{\xi^j\}$  of  $\Sigma$ ,  $\{x^j\} \subset \partial B(p, r)$  and  $\{x_i^j = \phi_i(x^j)\}$  in a way similar to the above construction of  $f_i$  in (4.1):

$$\tilde{f}_i = \tilde{f}_{\epsilon_\ell, r, i} = \frac{1}{\beta_{\Sigma_p}(\epsilon_\ell)} \sum_j \frac{1}{r} d(x_i^j, \cdot) : M_i^4 \rightarrow \mathbb{R}.$$

Similarly to (4.2), we have

$$\lim_{\ell \rightarrow \infty} \lim_{r \rightarrow 0} \lim_{i \rightarrow \infty} \tilde{f}_i \circ \phi_i \circ \psi_r = \tilde{f}.$$

Now take a local maximum point  $\hat{p}_i \in S_r^i$  of the function  $\tilde{f}_i$  restricted to  $S_r^i$  such that  $\hat{p}_i \rightarrow p$ . This is possible because  $\tilde{f}$  takes a strictly local maximum at  $o_p$  and  $(\frac{1}{r}S_r^i, \tilde{p}_i) \rightarrow (K_1(\Sigma), o_p)$  as  $i \rightarrow \infty$ ,  $r \rightarrow 0$ , where  $K_1(\Sigma)$  denotes the unit subcone of  $K(\Sigma)$ . Let  $\delta_i$  be the maximum distance from  $\hat{p}_i$  to the critical point set of  $d_{\hat{p}_i}$  on  $B(p_i, r)$ . Note that

$\delta_i \rightarrow 0$  since  $\hat{p}_i \rightarrow p$ . Let  $(Y, y_0)$  be any limit of  $(\frac{1}{\delta_i}M_i^4, \hat{p}_i)$ . By the splitting theorem,  $Y$  is isometric to a product  $Y_0 \times \mathbb{R}$ . Let  $q_i \in B(p_i, r)$  be a critical point of  $d_{\hat{p}_i}$  with  $d(\hat{p}_i, q_i) = \delta_i$ . We may assume that  $q_i$  converges to a point  $z_0 \in Y$  under the convergence  $(\frac{1}{\delta_i}M_i^4, \hat{p}_i) \rightarrow (Y, y_0)$ . The fact that  $z_0$  is a critical point of  $d_{y_0}$  yields  $\pi_2(y_0) = \pi_2(z_0)$ , where  $\pi_2 : Y_0 \times \mathbb{R} \rightarrow \mathbb{R}$  is the projection. For  $a \gg 1$ , take  $b_i \rightarrow \infty$  such that  $g_i(\hat{p}_i) = a$  for  $g_i := \tilde{f}_i - b_i$ . We may assume that  $g_i$  converges to a function  $g_\infty$  on  $Y$ . Since  $g_i$  also takes a local maximum on  $S_r^i$  at  $\hat{p}_i$ , it follows from construction that  $g_\infty$  takes a maximum on  $Y_0 \times \{\pi_2(y_0)\}$  at  $y_0$ . Now the argument of Lemma 3.5 in [43] yields  $\dim Y \geq 4$ , and the theorem holds.

The proof for the case that  $\Sigma = [0, \ell]$ ,  $\ell < \pi$  or the case that  $\dim X = 2$ ,  $\Sigma_p = [0, \pi]$  is similar to the above argument.

Finally let us consider the case that  $\Sigma = [0, \pi]$ , that is,  $(K_p, o_p) = (\mathbb{R}^2 \times [0, \infty), 0)$ . Take  $v_{1,r}, v_{2,r} \in \mathbb{R}^2 \times 0$  with  $d(o_p, v_{\alpha,r}) = \tau_r$ , and  $\angle(v_{1,r} o_p v_{2,r}) = \pi/2$ . Set  $w_{\alpha,r}^i = \phi_i \circ \psi_r(v_{\alpha,r})$  as before, and consider the two-dimensional submanifold

$$S_r^i := \{d_{w_{1,r}^i} = d_{w_{1,r}^i}(p_i)\} \cap \{d_{w_{2,r}^i} = d_{w_{2,r}^i}(p_i)\} \cap \{d_{p_i} \leq r\}.$$

Then we complete the proof by a similar argument.  $\square$

In the rest of this section, we consider the remaining case that  $\pi > \text{diam}(\Sigma_p) > \pi/2$ . From now on, we assume  $n = 4$ . In view of Lemma 4.8 and [43], we may assume  $\dim X = 3$ . For  $\xi, \eta \in \Sigma_p$  with  $d(\xi, \eta) = \text{diam}(\Sigma_p)$ , we put  $x = \exp r\xi'$ ,  $y = \exp r\eta'$ , where  $\xi', \eta'$  are sufficiently close to  $\xi$  and  $\eta$  respectively, and  $r$  is sufficiently small. Take  $0 < \epsilon_1 \ll \epsilon \ll r \leq \text{const}_p$ , and put  $w := \exp(\epsilon\xi') \in px$ ,

$$U_{\epsilon_1}(p, w) := B(w, \epsilon_1) \cap \partial B(p, \epsilon).$$

We may assume that for some fixed constant  $\delta < \text{diam}(\Sigma_p) - \pi/2$

- (1)  $\tilde{\angle} xzy > \pi/2 + \delta$  for any  $z \in B(p, \epsilon)$ ;
- (2)  $\angle pzy - \tilde{\angle} pzy < \tau(r)$  for any  $z \in B(p, \epsilon)$ ;
- (3)  $\angle pzy \leq \pi/2 - \delta$  for all  $z \in U_{\epsilon_1}(p, w)$ .

For  $x_i, y_i \in M_i^4$  with  $x_i \rightarrow x$ ,  $y_i \rightarrow y$ , we have  $\tilde{\angle} x_i z_i y_i > \pi/2 + \delta$  for every  $z_i \in B(p_i, \epsilon)$  and large  $i$ . Let  $w_i \in M_i^4$  be such that  $w_i \rightarrow w$ , and

$$U_{\epsilon_1}(p_i, w_i) := \partial B(p_i, \epsilon) \cap B(w_i, \epsilon_1).$$

Applying Lemma 4.8 to the convergence  $(M_i^4, w_i) \rightarrow (X^3, w)$ , we have sequences  $\delta_i \rightarrow 0$  and  $\hat{w}_i \rightarrow w$  satisfying

- (a)  $B(w_i, \epsilon_1) \simeq B(\hat{w}_i, R\delta_i)$  for every  $R \geq 1$  and large  $i$ ;
- (b) for any limit  $(Y, y_0)$  of  $\frac{1}{\delta_i}(M_i^4, \hat{w}_i)$ , we have  $\dim Y = 4$ .

Note that  $Y$  is isometric to a product  $\mathbb{R} \times Y_0$ . By Lemma 4.3,  $\dim Y(\infty) \geq 2$ . It follows from Proposition 2.4 that the dimension of the soul  $S_0$  of

$Y_0$  is 0 or 1. Since  $Y_0$  has no boundary, the generalized soul theorem in [43] implies that

$$B(S_0, R; Y_0) \simeq D^3 \quad \text{or} \quad S^1 \times D^2$$

for large  $R > 0$ .

**Lemma 4.9.** *We have*

$$\begin{aligned} U_{\epsilon_1}(p_i, w_i) &\simeq B(S_0, R; Y_0) \\ &\simeq D^3 \quad \text{or} \quad S^1 \times D^2. \end{aligned}$$

*Proof.* Let  $R_1 \gg R$  be large numbers, and let  $q_i$  be a point on  $p_i \hat{w}_i$  with  $d(q_i, \hat{w}_i) = \delta_i R_1$ . Let  $q \in Y$  be the limit of  $q_i$  under the convergence  $(\frac{1}{\delta_i} M_i^4, \hat{w}_i) \rightarrow (Y, y_0)$ . From the standard Morse theory for distance functions together with the proof of Lemma 4.8,

$$\begin{aligned} U_{\epsilon_1}(p_i, w_i) &\simeq U_{\epsilon_1}(p_i, \hat{w}_i) \\ &\simeq U_{\delta_i R}(p_i, \hat{w}_i) \\ &\simeq U_{\delta_i R}(q_i, \hat{w}_i). \end{aligned}$$

It should be noted here that to show the second  $\simeq$  above, we actually need to take a smooth approximation  $\tilde{d}_{p_i}$  of  $d_{p_i}$ . Since this is only a technical point, we omit the detail of this argument.

Now consider the convergence  $(\frac{1}{\delta_i} M_i^4, \hat{w}_i) \rightarrow (Y, y_0)$  and note that  $(d_{y_0}, d_q)$  is regular near  $(d_{y_0}, d_q)^{-1}(R, R_1)$ . It follows from Theorem 1.6 that

$$\begin{aligned} U_{\delta_i R}(q_i, \hat{w}_i) &\simeq U_R(q, y_0) \\ &\simeq B(y_0, R; Y_0) \\ &\simeq B(S_0, R; Y_0). \end{aligned}$$

This completes the proof.  $\square$

To complete the proof of Theorem 4.1, we need to determine the topology of  $B(p_i, r)$ . This is based on the following lemma.

**Lemma 4.10.** *There exist a positive number  $\epsilon$  independent of  $i$  and a unit vector field  $V_i$  on  $B(p_i, \epsilon)$  satisfying:*

- (1) *The flow curves of  $V_i$  are gradient-like for  $d_{y_i}$ ;*
- (2) *Let  $h_{z_i}^i(t)$  denote the flow curve of  $V_i$  starting from a point  $z_i$ .*

*Then we have*

- (a)  *$h_{z_i}^i(t)$  is transversal to  $\partial B(p_i, \epsilon)$  for every  $z_i \in U_{\epsilon_1}(p_i, w_i)$ ;*
- (b)

$$|\angle(V_i(h_{z_i}^i(t)), (p_i)'_{h_{z_i}^i(t)}) - \angle(y'_{\phi_i(z_i)}, p'_{\phi_i(z_i)})| < \tau(\epsilon, 1/i),$$

*for every  $z_i \in U_{\epsilon_1}(p_i, w_i) - U_{\epsilon_1/2}(p_i, w_i)$ .*

*Proof.* We construct a local gradient-like vector field for  $d_{y_i}$  on a small neighborhood of each point of  $B(p_i, \epsilon)$  and use a partition of unity to get a global one.

It is easy to construct a vector field  $W_i$  on a small neighborhood  $A_i$  of a broken geodesic  $w_i p_i \cup p_i y_i$  so as to satisfy (1), (2)-(a) and that the flow curve of  $V_i$  through  $w_i$  passes  $p_i$ . Extend  $W_i$  on  $B(p_i, \epsilon_2)$ ,  $\epsilon_2 = 10^{-10} \epsilon_1$  so as to satisfy (1). For each  $q_i \in B(p_i, \epsilon) - A_i - B(p_i, \epsilon_2)$ , let  $v_{q_i}$  denote a unit vector at  $q_i$  tangent to a minimal geodesic from  $q_i$  to  $y_i$ . The property (2) stated in the preceding paragraph of Lemma 4.9 yields that

$$(4.3) \quad |\angle(v_{q_i}, (p_i)'_{q_i}) - \angle(y'_q, p'_q)| < \tau(\epsilon, 1/i),$$

where  $q$  is a limit point of  $q_i$ . Let  $V_{q_i}$  be a smooth extension of  $v_{q_i}$  to a small neighborhood of  $q_i$ . By patching  $\{V_{q_i}\}$  with a partition of unity, we construct a vector field  $\tilde{W}_i$  on  $B(p_i, \epsilon) - A_i - B(p_i, \epsilon_2)$  satisfying (4.3) in place of  $v_{q_i}$ . To obtain the required vector field, it suffices to patch  $W_i$  and  $\tilde{W}_i$ .  $\square$

**Lemma 4.11.**

$$B(p_i, \epsilon) \simeq U_{\epsilon_1}(p_i, w_i) \times I.$$

*Proof.* Let  $\ell_i := p_i w_i \cup p_i y_i$  and consider a smooth approximation  $f_i$  of the function  $d(\ell_i, \cdot)$ , which is regular near  $f_i^{-1}(\epsilon_1)$  for some small  $\epsilon_1$  independent of  $i$ . Let  $S_i = B(p_i, \epsilon) \cap \{f_i = \epsilon_1\}$ . By using Lemma 4.10, one can construct a gradient-like vector field  $W_i$  for  $d_{y_i}$  on  $B_i := \{f_i \leq \epsilon_1\} \cap B(p_i, \epsilon)$  such that  $W_i$  is tangent to  $S_i$ . To do this, it suffices to take the tangential component of the vector field  $V_i$  to  $S_i$  in the orthogonal decomposition of  $V_i$ , where  $V_i$  is the vector field given in Lemma 4.10. Thus we have  $B_i \simeq U_{\epsilon_1}(p_i, w_i) \times I$ . Since  $f_i$  is regular on  $B(p_i, \epsilon) - \text{int } B_i$  and the gradient flow of it is transversal to  $\partial(B(p_i, \epsilon) - \text{int } B_i)$ , we conclude that  $B(p_i, \epsilon) \simeq B_i$ .  $\square$

*Proof of Theorem 4.1 in the case of  $\pi/2 < \text{diam}(\Sigma_p) < \pi$ .* By the previous lemma together with the topological assumption on  $B(p_i, r)$ ,  $B(p_i, \epsilon)$  is homeomorphic to  $S^1 \times D^3$ . Let  $\Gamma_i = \pi_1(B(p_i, \epsilon))$ , and  $\tilde{B}(p_i, \epsilon)$  the universal cover of  $B(p_i, \epsilon)$ . For a point  $\tilde{p}_i \in \tilde{B}(p_i, \epsilon)$  over  $p_i$ , put

$$\delta_i := \min_{\gamma \in \Gamma_i - \{1\}} d(\gamma \tilde{p}_i, \tilde{p}_i),$$

and choose a sequence  $\tau_i \rightarrow 0$  satisfying

$$\lim_{i \rightarrow \infty} \frac{\tau_i}{\max\{\delta_i, d_{GH}(B(p_i, \epsilon), B(p, \epsilon))\}} = \infty.$$

Let  $(Z, z_o, G)$  be any limit of  $(\frac{1}{\tau_i} \tilde{B}(p_i, \epsilon), \tilde{p}_i, \Gamma_i)$  with respect to the pointed equivariant Gromov-Hausdorff topology (see [20] for instance for the definition). Note that  $G$  is a lie group ([21]). By the choice of  $\tau_i$ ,  $\Gamma_i$  collapses, that is,  $\dim G > 0$ . Since  $\dim Z/G = 3$ , it follows that  $\dim Z = 4$ . Since  $Z$  is simply connected and  $\dim Z(\infty) \geq 2$ , the

soul of  $Z$  must be a point, and therefore Generalized Soul Theorem 2.6 implies that  $Z$  is homeomorphic to  $\mathbb{R}^4$ .

Let  $(W, w_0, \Gamma_\infty)$  be any limit of  $(\frac{1}{\delta_i} \tilde{B}(p_i, \epsilon), \tilde{p}_i, \Gamma_i)$  with respect to the pointed equivariant Gromov-Hausdorff topology. The previous argument shows  $\dim Y = 4$ . Moreover  $\Gamma_i$  does not collapse under this convergence, and therefore  $\Gamma_\infty \simeq \Gamma_i \simeq \mathbb{Z}$ . As before, since  $Y$  is simply connected and  $\dim Y(\infty) \geq 2$ , Generalized Soul Theorem 2.6 and Proposition 2.4 imply  $Y \simeq \mathbb{R}^4$ . Hence  $Y/\Gamma_\infty$ , the limit of  $(\frac{1}{\delta_i} M_i^4, p_i)$ , has a soul  $\simeq S^1$ . Corollary 2.7 then yields that  $B(p_i, R\delta_i) \simeq S^1 \times D^3$  for large  $R > 0$  and  $i$ . This completes the proof of Theorem 4.1.  $\square$

**Corollary 4.12.** *If a sequence of pointed 4-dimensional complete Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  converges to a pointed 3-dimensional complete Alexandrov space  $(X^3, p)$ , then there exists a positive number  $r = r_p$  depending only on  $p$  such that  $B(p_i, r)$  is homeomorphic to either  $D^4$  or  $S^1 \times D^3$  for sufficiently large  $i$ .*

*Proof.* Applying Theorem 4.1 to the convergence  $(M_i^4, p_i) \rightarrow (X^3, p)$ , we have sequences  $\delta_i \rightarrow 0$  and  $\hat{p}_i \in M_i$  such that

- (1)  $\hat{p}_i \rightarrow p$  under the convergence  $M_i^4 \rightarrow X^3$ ;
- (2)  $B(p_i, r)$  is homeomorphic to  $B(\hat{p}_i, R\delta_i)$  for every  $R \geq 1$  and large  $i \geq i(R)$ ;
- (3) for a limit  $(Y, y_0)$  of  $(\frac{1}{\delta_i} M_i, \hat{p}_i)$ , we have  $\dim Y = 4$ .

Since  $Y(\infty)$  has dimension at least two (Lemma 4.3), the soul of  $Y$  has dimension at most one (Proposition 2.4). Thus Corollary 2.7 implies the conclusion.  $\square$

## 5. GEOMETRY OF ALEXANDROV THREE-SPACES

From Theorem 4.1, it is significant in the first step to analyze collapsing of 4-manifolds to 3-dimensional Alexandrov spaces, which will be discussed in Sections 6, 7, 8 and 9. In this section, we investigate the structure near singular points of 3-dimensional Alexandrov spaces with curvature bounded below, and obtain two results: One is about the structure of the essential singular point set of a 3-dimensional Alexandrov space in terms of quasigeodesics (Proposition 5.7), and the other is about the existence of a collar neighborhood of 3-dimensional Alexandrov spaces with nonempty boundary (Theorem 5.14).

We first prepare some material on extremal subsets (see [38] for the details). By definition, a closed subset  $F$  in an Alexandrov space  $X$  with curvature bounded below is called *extremal* if the following holds: For any  $p \in X - F$ , consider the distance function  $d_p = d(p, \cdot)$ , and let  $q \in F$  be a local minimum point for the restriction  $d_p|_F$ . Then  $q$  is infinitesimally a local minimum of  $d_p$ , namely

$$\limsup_{q_i \in X \rightarrow q} \frac{d_p(q_i) - d_p(q)}{d(q_i, q)} \leq 0.$$

or equivalently,

$$\max_{\eta \in \Sigma_q} \min_{\xi \in p'_q} \angle(\xi, \eta) \leq \pi/2.$$

If  $F$  is a point,  $F$  is extremal if and only if it is an extremal point in the sense of Section 1.

**Example 5.1.** (1)  $\partial X$  is an extremal subset of  $X$ ;  
(2)  $\mathbb{R} \times o$  is an extremal subset of  $\mathbb{R} \times K(S_\ell^1)$ , where  $S_\ell^1$  denotes the circle of length  $\ell \leq \pi$  and  $o$  is the vertex of the cone  $K(S_\ell^1)$ .

Now consider an Alexandrov space  $\Sigma$  with curvature  $\geq 1$ . A closed set  $\Omega \subset \Sigma$  is called *extremal* if it satisfies the above condition together with

- (1)  $\Sigma = B(\Omega, \pi/2)$ ;
- (2)  $\text{diam}(\Sigma) \leq \pi/2$  if  $\Omega$  is either a point or empty.

**Example 5.2.** (1) Let  $\Sigma$  be the spherical suspension over a circle of length  $\leq \pi$ . Then the set consisting of the two vertices of  $\Sigma$  is extremal;  
(2) Let  $\xi_1, \xi_2$  and  $\xi_3$  be three points on the unit sphere  $S^2(1)$  with  $d(\xi_1, \xi_i) = \pi/2$ , ( $i = 2, 3$ ), and  $d(\xi_2, \xi_3) \leq \pi/2$ . Let  $D(\Delta)$  denote the double of the triangular region bounded by  $\Delta\xi_1\xi_2\xi_3$ . Then any nonempty subset of  $\{\xi_1, \xi_2, \xi_3\}$  is extremal in  $D(\Delta)$ .

**Lemma 5.3** ([38]).  *$F \subset X$  is extremal if and only if  $\Sigma_p(F) \subset \Sigma_p$  is extremal for any  $p \in F$ , where the space of directions  $\Sigma_p(F)$  of  $F$  at  $p$  is defined as the set of directions  $\xi \in \Sigma_p$  such that for some sequence  $x_n \in F$  with  $x_n \rightarrow p$ ,  $(x_n)'_p$  converges to  $\xi$ .*

Here we present some information on the essential singular point set in a 3-dimensional Alexandrov space. First we need

**Lemma 5.4.** *Let  $\Sigma$  be a compact Alexandrov space with curvature  $\geq 1$  and with boundary. Then there exists at most one extremal point of  $\text{int } \Sigma$ .*

*If  $\text{int } \Sigma$  has the unique extremal point, then it must be the unique maximum point of the distance function from the boundary  $\partial\Sigma$ .*

*Proof.* Note that  $f = d(\partial\Sigma, \cdot)$  is a strictly concave function on  $\Sigma$  ([35]), and therefore  $f$  has a unique maximum point. From the concavity, it is obvious to see that any non-maximum point of  $\text{int } \Sigma$  has the space of directions whose diameter is greater than  $\pi/2$ .  $\square$

**Lemma 5.5.** *Let  $\mathcal{C}$  be a closed subset of the essential singular point set  $ES(X^3)$  of a 3-dimensional Alexandrov space  $X^3$  satisfying the following:*

- (1) *For any  $p \in \mathcal{C}$ , either  $\Sigma_p(\mathcal{C})$  contains at least two elements, or it consists of only a point. In the latter case,  $p$  must be an extremal point of  $X^3$ ;*

(2)  $\mathcal{C} \cap \partial X^3$  is either empty or consists of components of  $\partial X^3$ .

Then  $\mathcal{C}$  is extremal.

*Proof.* Obviously for any  $p \in \mathcal{C}$ , every direction  $\xi \in \Sigma_p(\mathcal{C})$  is an essential singular point of  $\Sigma_p(X^3)$ . Therefore by Example 5.1 and Lemma 5.4, we may assume that  $\mathcal{C} \subset \text{int } X^3$ . By Lemma 5.3, it suffices to show that for any  $p \in \mathcal{C}$ ,  $\Sigma_p(\mathcal{C})$  is an extremal subset of  $\Sigma_p(X^3)$ . From assumption, it suffices to consider the case when  $\Sigma_p(\mathcal{C})$  contains distinct elements  $\xi_1$  and  $\xi_2$ . Then we have

$$(5.1) \quad L(\Sigma_{\xi_i}(\Sigma_p(X^3))) \leq \pi, \quad i = 1, 2.$$

If  $\text{diam}(\Sigma_p) \leq \pi/2$ , clearly  $\Sigma_p(\mathcal{C})$  is extremal. If  $\text{diam}(\Sigma_p) > \pi/2$ , then  $\xi_1$  and  $\xi_2$  realize the diameter of  $\Sigma_p$ , because any other point  $\xi_3$  from  $\xi_1, \xi_2$  does not satisfy (5.1). In view of (5.1), the Alexandrov convexity implies that  $B(\xi_1, \pi/2)$  and  $B(\xi_2, \pi/2)$  cover  $\Sigma_p$ , and hence  $\Sigma_p(\mathcal{C})$  is extremal.  $\square$

*Remark 5.6.* In Lemma 5.5, the two conditions on  $\mathcal{C}$  are essential. For instance, let  $\Sigma^2$  be a compact Alexandrov surface with curvature  $\geq 1$  such that

- (1) the radius of  $\Sigma^2$  is less than or equal to  $\pi/2$ ;
- (2)  $\Sigma^2$  contains at most one essential singular point;
  - (a) if  $\Sigma^2$  has no essential singular point, then  $\text{diam}(\Sigma^2) > \pi/2$ ;
  - (b) if  $\Sigma^2$  has an essential singular point  $\xi$ , then there is an  $\eta$  with  $d(\xi, \eta) > \pi/2$ .

For  $X^3 := K(\Sigma^2)$ ,  $ES(X^3)$  is not extremal because the extremal condition does not hold at the vertex of the cone  $K(\Sigma^2)$ .

Let  $X$  be a complete Alexandrov space with curvature  $\geq \kappa$ . For a curve  $\gamma : [a, b] \rightarrow X$  and a point  $p \in X$ , a curve  $\tilde{\gamma}$  on the  $\kappa$ -plane  $M_\kappa^2$  is called a *development* of  $\gamma$  from  $p$  if  $d(\tilde{\gamma}(t), \tilde{p}) = d(\gamma(t), p)$  for some point  $\tilde{p}$ . The curve  $\gamma : [a, b] \rightarrow X$  is called a *quasigeodesic* (see [38]) if for any point  $p \in X$ , the development  $\tilde{\gamma}$  of  $\gamma$  from  $p$  defines a convex subset bounded by  $\tilde{p}\tilde{\gamma}(t_1)$ ,  $\tilde{p}\tilde{\gamma}(t_2)$  and  $\tilde{\gamma}([t_1, t_2])$  for any  $a \leq t_1 < t_2 \leq b$ .

**Proposition 5.7.** *Let  $X^3$  be a 3-dimensional complete Alexandrov space with curvature bounded below, and let  $\mathcal{C}$  be a subset of  $ES(\text{int } X^3)$  satisfying the condition (1) of Lemma 5.5. Then  $\mathcal{C}$  has the structure of a finite metric graph such that*

- (1) every vertex of  $\mathcal{C}$  as a graph except the endpoints has order three;
- (2) every subarc of  $\mathcal{C}$  is a quasigeodesic.

*Proof.* By Lemma 5.5,  $\mathcal{C}$  is extremal. It follows that  $\mathcal{C}$  is locally connected by quasigeodesics ([38]) and that there exists a quasigeodesic starting at  $p$  in a direction  $\xi \in \Sigma_p(\mathcal{C})$ . The uniqueness of such a quasigeodesic follows from the argument in the proof of Assertion 6.12. Note that  $\Sigma_p$  has at most three essential singular points (see Appendix in [43]). This completes the proof.  $\square$

**Example 5.8.** Let  $\Sigma$  be a compact Alexandrov surface with curvature  $\geq 1$  and with exactly three essential singular points, say  $p_1, p_2$  and  $p_3$ . We consider the spherical suspension  $X^3 = \Sigma \times_{\sin t} [0, \pi]$  of  $\Sigma$ , where we make an identification  $\Sigma = \Sigma \times \{\pi/2\}$ .  $X$  has the essential singular point set consisting of three minimal geodesic segments joining the two poles of  $X^3$  through  $p_i$ .

**Example 5.9.** We construct a 3-dimensional complete open Alexandrov space  $X^3$  with nonnegative curvature whose essential singular point set consists of a countable set with a limit as follows: First we construct a noncompact convex body  $E$  in the  $(x, y, z)$ -space as follows. Let  $C$  be a compact convex “polygon” on the  $(x, y)$ -plane with countable edges such that a sequence  $p_i$  of consecutive vertices of  $\partial C$  converges to a point  $p \in \partial C$ . We denote by  $\{e_i\}$  the line segment from  $p_{i-1}$  to  $p_i$  in  $\partial C$ . For a point  $q_i \in \mathbb{R}^2 - C$  sufficiently close to the midpoint of  $e_i$ , let  $\ell_i$  be the geodesic ray in  $\mathbb{R}^3$  starting from  $(q_i, 1)$  in the direction to the positive  $z$ -axis. Let  $E_i$  be the minimal convex set containing  $C \times [0, \infty)$  and  $\ell_1 \cup \ell_2 \cup \dots$ . If  $d(q_i, e_i)$  is sufficiently small, then  $p_i \times [0, \infty)$  is contained in  $\partial E$ . Finally we take the double  $X^3 := D(E)$ .  $X^3$  has nonnegative curvature, and  $ES(X^3)$  consists of the sequence  $p_i, i = 1, 2, \dots$  with the limit  $p$ .

Example 5.9 shows that in Proposition 5.7 one cannot drop the condition (1) in Lemma 5.5.

Next we turn to the other subject of this section, the existence of a collar neighborhood of the boundary of an Alexandrov space with curvature bounded below.

**Proposition 5.10.** *Let  $X^n$  be an Alexandrov space with curvature bounded below and with nonempty boundary. Then  $\partial X^n$  has a collar neighborhood, i.e., an open neighborhood of  $\partial X^n$  homeomorphic to  $\partial X^n \times [0, 1)$ .*

*Proof.* This is done by induction on  $n$ . The case of  $n = 1$  is clear. By [5], it suffices to show that  $\partial X$  is locally collared. For any  $p \in \partial X$ ,  $\partial \Sigma_p$  has a collar neighborhood. This implies that  $\partial K_p$  has a collar neighborhood. Thus there is an  $\epsilon > 0$  such that  $B(p, \epsilon) \cap \partial X$  has a collar neighborhood.  $\square$

Since the collar neighborhood given in Proposition 5.10 is only topological, it is not enough for our purpose. We need a metric collar neighborhood at least when  $\partial X$  is compact.

We put for  $\epsilon > 0$

$$X_\epsilon = \{x \in X \mid d(x, \partial X) \geq \epsilon\}.$$

**Conjecture 5.11.** Let  $n$  be any positive integer, and let  $X^n$  be an  $n$ -dimensional Alexandrov space with curvature bounded below and with



nonempty compact boundary. Then there is a positive number  $\epsilon$  such that  $X^n - X_\epsilon^n$  provides a collar neighborhood of  $\partial X^n$ .

The author is not certain if the method in [5] can be applied to solve Conjecture 5.11 affirmatively. Here is a conjecture related with Conjecture 5.11.

**Conjecture 5.12** ([38]). If  $X^n$  is an Alexandrov space with curvature  $\geq \kappa$ , then so is the boundary  $\partial X^n$  (if it is nonempty) with respect to the length metric induced from  $X^n$ .

Conjecture 5.11 will be true if we come to know Conjecture 5.12 to be true:

**Observation 5.13.** *If Conjecture 5.12 is true, then Conjecture 5.11 is also true.*

*Proof.* Let  $Y^n$  denote the gluing

$$Y^n := X^n \cup_{\partial X^n} \partial X^n \times [0, \infty).$$

By the assumption,  $\partial X^n \times [0, \infty)$  is an Alexandrov space with curvature bounded below. It follows from [39] that  $Y^n$  is also an Alexandrov space with curvature bounded below. Put  $Z := \partial X^n \times 1 \subset Y^n$  and consider

$$f = d(Z, \cdot) : Y^n \rightarrow \mathbb{R}.$$

Obviously  $f$  is regular on  $\partial X^n \times 0$ . Observation 5.13 immediately follows from [35].  $\square$

**Theorem 5.14.** *Conjecture 5.11 is true for  $n = 3$ .*

*Proof.* First note that each point  $p$  of  $\partial X^3$  has a small spherical neighborhood homeomorphic to  $\mathbb{R}_+^3$ . Consider the convergence  $(\frac{1}{r}X^3, p) \rightarrow (K_p, o_p)$  as  $r \rightarrow 0$ . Since  $\Sigma_p - (\Sigma_p)_\epsilon$  provides a collar neighborhood of  $\partial \Sigma_p$  for small  $\epsilon$ , it follows that  $K_p - (K_p)_\epsilon$  also provides a collar neighborhood of  $\partial K_p$ . Note that  $d_{\partial K_p}$  and  $(d_{\partial K_p}, d_{o_p})$  are regular on  $\{0 < d_{\partial K_p} \leq \epsilon\}$  and  $\{0 < d_{\partial K_p} \leq \epsilon, d_{o_p} \geq 1/2\}$  respectively. This ensures the existence of a positive number  $r$  such that  $B(p, r) - X_\epsilon^3$  provides a collar neighborhood of  $\partial X^3 \cap B(p, r)$  for  $\epsilon \ll r$ . We take a finite covering  $\{B(p_j, r_j/2)\}_{1 \leq j \leq N}$  of  $\partial X^3$ , where  $\epsilon_j \ll r_j$  are chosen for  $p_j$  as above. Let  $K$  be a triangulation of  $\partial X^3$  by Lipschitz curves, and take sufficiently small  $\epsilon$  compared with  $\min\{\epsilon_j\}$  and the sizes of the simplices of  $K$ . Let  $K^i$  denote the  $i$ -skeleton of  $K$ . For each  $p \in K^0$ , choose a minimal geodesic  $\gamma_p$  from  $p$  to  $\partial X_\epsilon$ . The disjoint union of  $\{\gamma_p\}_{p \in K^0}$  provides a collar of  $K_0$ . For each edge  $e \in K^1$ , choose a Lipschitz curve  $e_1$  on  $\partial X_\epsilon$  such that

- (1) the union of  $e$ ,  $e_1$ ,  $\gamma_p$  and  $\gamma_q$  bounds a 2-disk in  $X^3 - \text{int } X_\epsilon^3$ , denoted  $D_e$ , giving a collar of  $e$ , satisfying  $\text{int } D_e \subset \text{int } X^3 - X_\epsilon^3$ , where  $p$  and  $q$  are the endpoints of  $e$ ;
- (2) if two edges  $e$  and  $e'$  of  $K$  meet at a vertex  $p$ , then  $D_e \cap D_{e'} = \gamma_p$ .

This is possible because of the local collar structure mentioned above. Now for every 2-simplex  $\Delta^2$  of  $K$ , let  $D_{\partial\Delta^2}$  be the collar of  $\partial\Delta^2$  constructed above. Let  $\hat{\Delta}^2$  denote a disk domain of  $\partial X_\epsilon^3$  bounded by  $D_{\partial\Delta^2} \cap \partial X_\epsilon$ . Since the union of  $\Delta^2$ ,  $D_{\partial\Delta^2}$  and  $\hat{\Delta}^2$  is homeomorphic to  $S^2$  and locally flat, by the generalized Schoenflies theorem, it bounds a closed domain homeomorphic to  $D^3$ , which gives a collar of  $\Delta^2$  extending the collar structure given by  $D_{\partial\Delta^2}$ . Thus we obtain a collar structure on  $X^3 - X_\epsilon^3$ .  $\square$

## 6. COLLAPSING TO THREE-SPACES WITHOUT BOUNDARY-LOCAL CONSTRUCTION

Let a sequence of complete 4-dimensional pointed Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  converge to a 3-dimensional pointed Alexandrov space  $(X^3, p)$ . Throughout this section, we assume that  $p$  is an interior point of  $X^3$ . The purpose of this section is to construct an  $S^1$ -action on a small perturbation of  $B(p_i, r)$ , for a sufficiently small  $r > 0$  depending only on  $p$ .

By Fibration Theorem 1.2, we have a locally trivial  $S^1$ -bundle  $f_i : M_i' \rightarrow \text{int } B(p, 1) - S_{\delta_3}(X^3)$  which is  $\epsilon_i$ -approximation,  $\lim_{i \rightarrow \infty} \epsilon_i = 0$ , where  $M_i'$  is an open subset of  $M_i^4$  and  $\delta_3 > 0$  is sufficiently small.

Our first step is to prove that  $\partial B(p_i, r)$  is a Seifert fibred space over  $\partial B(p, r)$ .

Let  $\delta \ll r$  and consider  $B(q, \delta)$  for every  $q \in \partial B(p, r)$ . We take a point  $q_i \in \partial B(p_i, r)$  Gromov-Hausdorff close to  $q \in \partial B(p, r)$ .

**Lemma 6.1.** *There exist positive numbers  $r_p$  and  $c_q$  such that if  $r \leq r_p$  and  $\delta/r \leq c_q$ , then for some  $r_1 < r < r_2$  and  $\delta_1 < \delta < \delta_2$  sufficiently close to  $r$  and  $\delta$  respectively, there exists a homeomorphism  $h_i : (d_{p_i}, d_{q_i})^{-1}([r_1, r_2] \times [\delta_1, \delta_2]) \rightarrow T^2 \times [r_1, r_2] \times [\delta_1, \delta_2]$  which respects  $(d_{p_i}, d_{q_i})$ , that is,  $p_r \circ h_i = (d_{p_i}, d_{q_i})$ , where  $p_r$  is the projection to the factor  $[r_1, r_2] \times [\delta_1, \delta_2]$ . In particular,  $\partial B(p_i, r) \cap \partial B(q_i, \delta)$  is homeomorphic to  $T^2$ .*

*Proof.* Note that  $(d_p, d_q)$  is regular near  $(d_p, d_q)^{-1}(r, \delta)$  for  $\delta \ll r \leq \text{const}_p$ . By [35],  $(d_p, d_q)^{-1}(r, \delta) \simeq S^1$  and we have a homeomorphism  $h : (d_p, d_q)^{-1}([r_1, r_2] \times [\delta_1, \delta_2]) \rightarrow S^1 \times [r_1, r_2] \times [\delta_1, \delta_2]$  which respects  $(d_p, d_q)$  for some  $r_1 < r < r_2$  and  $\delta_1 < \delta < \delta_2$  sufficiently close to  $r$  and  $\delta$  respectively. Since  $(d_p, d_q)^{-1}([r_1, r_2] \times [\delta_1, \delta_2])$  does not meet  $S_{\delta_3}(X^3)$ , Fibration Theorem 1.2 together with a standard Morse theory for distance functions yields

$$\begin{aligned} & (d_{p_i}, d_{q_i})^{-1}([r_1, r_2] \times [\delta_1, \delta_2]) \\ & \simeq S^1\text{-bundle over } (d_p, d_q)^{-1}([r_1, r_2] \times [\delta_1, \delta_2]), \\ & \simeq (S^1\text{-trivial bundle over } S^1) \times [r_1, r_2] \times [\delta_1, \delta_2]. \end{aligned}$$

From the regularity of  $(d_{p_i}, d_{q_i})$ , we have the required homeomorphism.  $\square$

Recall that

$$U_\delta(p_i, q_i) := \partial B(p_i, r) \cap B(q_i, \delta),$$

for  $q_i \in \partial B(p_i, r)$ .

**Lemma 6.2.**  $U_\delta(p_i, q_i) \simeq S^1 \times D^2$  for sufficiently small  $r$ ,  $\delta \ll r$  and large  $i$ .

*Proof.* It follows from Lemma 4.9 that  $U_\delta(p_i, q_i)$  is homeomorphic to either  $S^1 \times D^2$  or  $D^3$  for any sufficiently large  $i$ . Hence the result follows from Lemma 6.1.  $\square$

**Proposition 6.3.** If  $p$  is an interior point of  $X^3$  and if  $\text{diam}(\Sigma_p) > \pi/2$ , then  $B(p_i, r) \simeq S^1 \times D^3$ .

*Proof.* This follows from Lemmas 6.2 and 4.11.  $\square$

Since  $S_\delta(\Sigma_p)$  is finite for every  $\delta > 0$ ,  $S_\delta(K_p)$  is the union of finitely many geodesic rays starting from  $o_p$ . In view of the convergence  $(\frac{1}{r}X^3, p) \rightarrow (K_p, o_p)$  as  $r \rightarrow 0$ , it is possible to take a small positive number  $\epsilon = \epsilon_p$  and a finite subset  $\{q^j\}_{j=1, \dots, k(p)}$  of  $\partial B(p, r) \cap S_{\delta_3}(X^3)$  for any sufficiently small  $r = r(p, \epsilon) > 0$  such that

- (1)  $\{U_{\epsilon r}(p, q^j)\}_j$  covers  $\partial B(p, r) \cap S_{\delta_3}(X^3)$ ;
- (2)  $(U_{\epsilon r}(p, q^j) - U_{\epsilon^{10}r}(p, q^j))$  does not meet  $S_\delta(X^3)$ ;
- (3)  $\{U_{\epsilon r}(p, q^j)\}_j$  is disjoint.

Since  $\Sigma_p$  is topologically regular,  $U_{\epsilon r}(p, q^j)$  is homeomorphic to  $D^2$  for a small  $\epsilon$ .

Let  $\xi_j \in S_{\delta_3}(\Sigma_p)$  be the direction with  $\angle(\xi_j, (q^j)'_p) < \tau(r)$ , and take  $q_i^j \in \partial B(p_i, r)$  with  $q_i^j \rightarrow q^j$ .

**Lemma 6.4.** Let  $\epsilon = \epsilon_p$  be as above. Then there exists a positive number  $r_p$  such that for every  $0 < r \leq r_p$  we have a three-dimensional submanifold  $U_{j,i}$  of  $M_i^4$ , a small perturbation of  $U_{\epsilon r}(p_i, q_i^j)$ , satisfying the following:

- (1)  $d_{GH}(U_{j,i}, U_{\epsilon r}(p, q^j)) \rightarrow 0$  as  $i \rightarrow \infty$ ;
- (2)  $U_{j,i}$  is a fibred solid torus over  $U_{\epsilon r}(p, q^j)$  such that
  - (a)  $\partial U_{j,i} = f_i^{-1}(\partial U_{\epsilon r}(p, q^j))$  and the fibre structure on  $\partial U_{j,i}$  induced from that of  $U_{j,i}$  is compatible to the  $S^1$ -bundle structure on  $\partial U_{j,i}$  defined by  $f_i$ ;
  - (b) the Seifert invariants of the singular fibre in  $U_{j,i}$  (if it exists) do not exceed

$$2\pi/L(\Sigma_{\xi_j}(\Sigma_p));$$

(3) Let

$$\partial B^{f_i}(p_i, r) := f_i^{-1}(\partial B(p, r) - \cup_j U_{\epsilon r}(p, q^j)) \bigcup (\cup_j U_{j,i}),$$

and let  $B^{f_i}(p_i, r)$  denote the closed domain bounded by  $\partial B^{f_i}(p_i, r)$  containing  $p_i$ . Then  $B^{f_i}(p_i, r) \simeq B(p_i, r)$ .

In particular,  $\partial B^{f_i}(p_i, r)$  is a Seifert fibred space over  $\partial B(p, r)$ .

*Proof.* First note that both  $d_{p_i}$ -flow curves and  $d_{q_i}$ -flow curves are transversal to  $f_i^{-1}(\partial U_{er}(p, q^j))$ . Hence using some  $d_{q_i}$ -flow curves starting from  $f_i^{-1}(\partial U_{er}(p, q^j))$ , we can extend  $f_i^{-1}(\partial U_{er}(p, q^j))$  to a three-dimensional submanifold  $\hat{U}_{j,i}$  such that

- (1)  $\partial \hat{U}_{j,i}$  is the disjoint union of  $f_i^{-1}(\partial U_{er}(p, q^j))$  and  $\partial U_{er/2}(p_i, q_i^j)$ ;
- (2)  $\hat{U}_{j,i} \simeq f_i^{-1}(\partial U_{er}(p, q^j)) \times I$  through  $d_{q_i}$ -flow curves ;
- (3)  $d_{p_i}$ -flow curves are transversal to  $\hat{U}_{j,i}$ ;
- (4) The Hausdorff distance between  $\hat{U}_{j,i}$  and  $U_{er}(p_i, q_i^j) - U_{er/2}(p_i, q_i^j)$  is less than  $\tau_i r$ , where  $\lim_{i \rightarrow \infty} \tau_i = 0$ .

We let  $U_{j,i}$  denote the union of  $\hat{U}_{j,i}$  and  $U_{er/2}(p_i, q_i^j)$ . It follows from (3) that  $B^{f_i}(p_i, r) \simeq B(p_i, r)$ , where  $B^{f_i}(p_i, r)$  is defined as above. By (1) and (2),  $U_{j,i} \simeq D^2 \times S^1$ . Therefore from construction, we only have to prove the conclusion (2)-(b) for  $U_{j,i}$  chosen in this way. Let  $V_{i,j}^r$  denote the compact domain consisting of the flow curves of  $d_{p_i}$  through  $U_{er}(p_i, q_i^j)$  and contained in  $A(p_i; r_1, r_2)$  for  $r_1 = 0.9r$  and  $r_2 = 1.1r$ . We may assume that  $V_{i,j}^r$  converges to a compact domain  $V_j^r$  containing  $U_{er/2}(p, q^j)$ . Since  $(\frac{1}{r}V_j^r, q^j)$  converges to  $\Omega_j := K(U_{er/2}(o_p, \xi_j)) \cap \{0.9 \leq d_{o_p} \leq 1.1\}$ , we have

$$d_{p.GH} \left( \left( \frac{1}{r}V_{i,j}^r, q_i^j \right), (\Omega_j, \xi_j) \right) < \mu_i^r/r + \tau(r),$$

where  $\lim_{i \rightarrow \infty} \mu_i^r = 0$  (for any fixed  $r$ ). Therefore for a sequence  $\nu_\alpha \rightarrow 0$  as  $\alpha \rightarrow \infty$ , we see

$$d_{p.GH} \left( \left( \frac{1}{\nu_\alpha r}V_{i,j}^r, q_i^j \right), (K_{\xi_j}(\Sigma_p) \times \mathbb{R}, (o_{\xi_j}, 0)) \right) < \mu_i^r/\nu_\alpha r + \tau(r)/\nu_\alpha + \tau(1/\alpha),$$

which converges to zero if we take a sequence depending on  $(\alpha, r, i)$  in such a way that  $\alpha \gg 1$  and  $\tau(r)/\nu_\alpha \ll 1$  and  $\mu_i^r/\nu_\alpha r \ll 1$ . Let  $\tilde{V}_{i,j}^r \rightarrow V_{i,j}^r$  be the universal cover and  $\Gamma_{i,j}$  the deck transformation group of it. Passing to a subsequence, we may assume that  $(\frac{1}{\nu_\alpha r}V_{i,j}^r, q_i^j, \Gamma_{i,j})$  converges to a triplet  $(Z, z_0, G)$  with respect to the pointed equivariant Gromov-Hausdorff convergence, where  $Z$  is a complete noncompact Alexandrov space with nonnegative curvature such that  $Z/G$  is isometric to  $K(S_{\ell_0}^1) \times \mathbb{R}$  with  $\ell_0 = L(\Sigma_{\xi_j}(\Sigma_p))$ . In a way similar to Section 4 of [43], we have

- (1)  $Z$  is isometric to a product  $K(S_\ell^1) \times \mathbb{R} \times \mathbb{R}$ ;
- (2)  $G$  is isomorphic to  $\mathbb{Z}_\mu \times \mathbb{R}$ , where  $\mu \leq 2\pi/\ell_0$  and

$$\mathbb{Z}_\mu \times \mathbb{R} \subset \text{Isom}(K(S_\ell^1)) \times \text{Isom}(\mathbb{R}) \subset SO(2) \times \mathbb{R},$$

from which we obtain a fibred solid torus structure on  $U_{er}(p_i, q_i^j)$  of type  $(\mu, \nu)$  for some  $\nu$ , compatible to the fibres of  $f_i$  on the boundary. From the construction of  $U_{j,i}$ , the fibre structure on  $U_{er}(p_i, q_i^j)$  defines

a fibre structure on  $U_{j,i}$  of the same type as  $U_{er}(p_i, q_i^j)$ , compatible to the fibres of  $f_i$  on the boundary.  $\square$

Let  $\tilde{f}_i : \partial B^{f_i}(p_i, r) \rightarrow \partial B(p, r)$  be the Seifert fibration given in Lemma 6.4,

**Proposition 6.5.** *There exists a positive number  $r_0 = r_0(p)$  such that for any  $0 < r \leq r_0$  and sufficiently large  $i$  the Seifert fibration  $\tilde{f}_i : \partial B^{f_i}(p_i, r) \rightarrow \partial B(p, r)$  satisfies that*

- (1) *the number of singular fibres is at most two;*
- (2) *for any singular fibre over a point  $q \in \partial B(p, r)$ , there is an essential singular point  $\xi$  of  $\Sigma_p$  with  $\angle(\xi, q'_p) < \tau(r)$  such that the Seifert invariants of the singular fibre do not exceed*

$$\frac{2\pi}{L(\Sigma_\xi(\Sigma_p))};$$

- (3)  *$\tilde{f}_i$  comes from some  $S^1$ -action on  $\partial B^{f_i}(p_i, r)$ .*

In view of Lemma 6.4, for the proof of Proposition 6.5, it suffices to check (1) and (3). We need the following

**Proposition 6.6.** *Suppose that a sequence of pointed complete orientable Riemannian 4-manifolds  $(M_i^4, p_i)$  collapses to a pointed Alexandrov 3-space  $(X^3, p)$  under  $K \geq -1$ . Then  $X^3$  is a topological manifold.*

*Proof.* In view of Stability Theorem 1.5, it suffices to show that  $\Sigma_p$  is homeomorphic to  $S^2$  or  $D^2$ . If  $p \in \partial X^3$ , then  $\Sigma_p$  is an Alexandrov surface with curvature  $\geq 1$  and with nonempty boundary, and hence it is homeomorphic to  $D^2$ . If  $p \in \text{int } X^3$ , then  $\Sigma_p$  is homeomorphic to either  $S^2$  or  $P^2$ . Suppose  $\Sigma_p \simeq P^2$ . By lemma 6.4, we have a Seifert fibration  $\tilde{f}_i : \partial B(p_i, r) \rightarrow \Sigma_p$ . Note that the number  $m_i$  of the singular orbits of  $\tilde{f}_i$  satisfies  $m_i \leq 1$ . Otherwise, the universal cover  $\tilde{\Sigma}_p$  of  $\Sigma_p$  would contain more than three essential singular points. This is a contradiction to the curvature condition that  $\tilde{\Sigma}_p$  has curvature  $\geq 1$  (see for instance Appendix in [43]). If  $m_i = 0$ , then  $\partial B^{f_i}(p_i, r)$  is an  $S^1$ -bundle over  $P^2$ , denoted  $P^2 \tilde{\times} S^1$ . This is a contradiction to Lemma 4.11. If  $m_i = 1$ , then  $\partial B^{f_i}(p_i, r)$  is homeomorphic to  $P^2 \tilde{\times} S^1$  or a prism manifold (see [32] for instance). This is also a contradiction to Lemma 4.11.  $\square$

*Proof of Proposition 6.5.* . By Corollary 4.12,  $B(p_i, r)$  is homeomorphic to either  $S^1 \times D^3$  or  $D^4$ . If  $B(p_i, r) \simeq S^1 \times D^3$ , [40], p.459, shows that the number  $m_i$  of singular fibres of  $\tilde{f}_i : \partial B^{f_i}(p_i, r) \simeq S^1 \times S^2 \rightarrow \partial B(p, r) \simeq S^2$  satisfies that  $m_i = 0$  or  $2$ . If  $B(p_i, r) \simeq D^4$ , then the Seifert bundle  $\tilde{f}_i : \partial B^{f_i}(p_i, r) \simeq S^3 \rightarrow \partial B(p, r) \simeq S^2$  has at most two singular fibres (see [32]). In either case,  $\tilde{f}_i$  comes from an  $S^1$ -action on  $\partial B^{f_i}(p_i, r)$  (see [32]). This completes the proof of Proposition 6.5.  $\square$

Our next step is to construct an  $S^1$ -action on  $B^{f_i}(p_i, r)$  extending  $\tilde{f}_i$ .

Let us begin with the following lemma, which easily follows from the convergence  $(\frac{1}{r}X^3, p) \rightarrow (K_p, o_p)$  and the finiteness of  $S_\delta(\Sigma_p)$  for any  $\delta > 0$ .

**Lemma 6.7.** *For any  $p \in X^3$ , let  $\epsilon = \epsilon_p > 0$  be as in Lemma 6.4. Then there exist a positive integer  $k = k(p)$  and  $r = r(p, \epsilon) > 0$  such that for some  $x^1, \dots, x^k \in \partial B(p, 10r)$ ,*

- (1)  $B(p, 10r) \cap S_{\delta_3}(X^3) \subset \bigcup_{j=1}^k B(px^j, \epsilon r)$ ;
- (2)  $(B(px^j, \epsilon r) - B(px^j, \epsilon^{10}r)) \cap A(p; r/100, 10r)$  does not meet  $S_{\delta_3}(X^3)$ ;
- (3)  $\{B(px^j, \epsilon r) \cap A(p; r/100, 10r)\}_j$  is disjoint.

For each  $1 \leq j \leq k$  and for  $\epsilon_1 \in (0, 2\epsilon]$ ,  $r_2 \leq r_1 \leq 2r$ , we consider

$$\begin{aligned} A_j(p; \epsilon) &:= B(px^j, \epsilon r) \cap B(p, 2r), \\ A_j(p; \epsilon_1; r_1) &:= A_j(p; \epsilon_1) \cap B(p, r_1) \\ A_j(p; \epsilon_1; r_2, r_1) &:= A_j(p; \epsilon_1) \cap A(p; r_2, r_1) \\ A^{f_i}(p_i; r_1, r) &:= \{x \in B^{f_i}(p_i, r) \mid d(p_i, x) \geq r_1\}. \end{aligned}$$

Fibration Theorem 1.2 enables us to take a closed domain  $A_j^{f_i}(p_i; \epsilon) \subset B(p_i, 2r)$  such that

- (1)  $A_j^{f_i}(p_i; \epsilon)$  converges to  $A_j(p; \epsilon)$  under the convergence  $M_i^4 \rightarrow X^3$ ;
- (2)  $\partial A_j^{f_i}(p_i; \epsilon) \cap \text{int } A(p_i; r/100, 2r)$  coincides with  $f_i^{-1}(\partial A_j(p; \epsilon) \cap \text{int } A(p; r/100, 2r))$ .

Take  $x_i^j \in \partial B(p_i, 10r)$  close to  $x^j \in \partial B(p, 10r)$ , and let  $\tilde{d}_{p_i x_i^j}$  be a smooth approximation of the distance function  $d_{p_i x_i^j}$ . We consider

$$\begin{aligned} A_j^i(p_i; \epsilon) &:= \{\tilde{d}_{p_i x_i^j} \leq \epsilon r\} \cap B(p_i, 2r), \\ A_j^i(p_i; \epsilon_1; r_2, r_1) &:= A_j^i(p_i; \epsilon_1) \cap A(p_i; r_2, r_1) \\ A_j^{f_i}(p_i; \epsilon; r_1, r) &:= A_j^{f_i}(p_i; r_1, r) \cap A_j^i(p_i; \epsilon). \end{aligned}$$

**Lemma 6.8.**  $A_j^{f_i}(p_i; \epsilon; r/100, r) \simeq (S^1 \times D^2) \times I$ .

*Proof.* By using the flow curves of a gradient vector field for  $\tilde{d}_{p_i x_i^j}$ , we have

$$A_j^{f_i}(p_i; \epsilon; r/100, r) \simeq A_j^i(p_i; \epsilon) \cap A_j^{f_i}(p_i; r/100, r).$$

By Lemma 6.2 together with the flow curves of a gradient-like vector field for  $d_{p_i}$ ,

$$A_j^i(p_i; \epsilon) \cap A_j^{f_i}(p_i; r/100, r) \simeq (S^1 \times D^2) \times I,$$

and hence  $A_j^{f_i}(p_i; \epsilon; r/100, r) \simeq (S^1 \times D^2) \times I$ . □

Now we construct an  $S^1$ -action on  $B^{f_i}(p_i, r)$  extending  $\tilde{f}_i$ . Let us first consider

*Case I.*  $B(p_i, r) \simeq S^1 \times D^3$ .

In this case, the number  $m_i$  of singular fibres of  $\tilde{f}_i : \partial B^{f_i}(p_i, r) \simeq S^1 \times S^2 \rightarrow \partial B(p, r) \simeq S^2$  satisfies that  $m_i = 0$  or  $2$ . If  $m_i = 0$ ,  $\tilde{f}_i$  is a trivial bundle and extends to a trivial bundle  $B^{f_i}(p_i, r) \rightarrow B(p, r)$  (compare Lemma 6.9 below). Thus we obtain a free  $S^1$ -action on  $B^{f_i}(p_i, r)$  whose orbit space is homeomorphic to  $B(p, r)$ , as the collapsing structure on  $B^{f_i}(p_i, r)$ .

Before considering the essential case  $m_i = 2$ , we need the following elementary lemma on extending  $S^1$ -actions. Probably this is obvious for specialist, but for the lack of references we shall give a proof.

Let  $\phi_{\mu, \nu}$  denote the  $S^1$ -action on the solid torus  $S^1 \times D^2$  given by the canonical fibred solid torus structure of type  $(\mu, \nu)$ . Namely,  $\phi_{\mu, \nu}$  comes from the standard  $\mathbb{Z}_\mu$ -action on  $S^1 \times D^2$  generated by

$$\tau_{\mu\nu}(e^{i\theta_1}, re^{i\theta_2}) = (e^{i(\theta_1 + \frac{2\pi}{\mu})}, re^{i(\theta_2 + \frac{2\nu}{\mu}\pi)}).$$

We consider the canonical orientation of  $S^1 \times D^2 \times I$ , and the orientation on  $\partial(S^1 \times D^2 \times I)$  as boundary.

**Lemma 6.9.** *Consider an  $S^1$ -action  $\varphi$  on  $(S^1 \times \partial D^2 \times I) \cup (S^1 \times D^2 \times \{0\})$  such that*

- $\varphi$  defines an  $S^1$ -action, denoted  $\phi_0$ , on  $S^1 \times D^2 \times \{0\}$ , equivariant to  $\phi_{\mu, \nu}$ ;
- $\varphi$  defines a free  $S^1$ -action on  $S^1 \times \partial D^2 \times I$ .

*Then the following holds:*

- (1)  $\varphi$  extends to a locally smooth  $S^1$ -action  $\psi$  on  $S^1 \times D^2 \times I$ ;
- (2) Any such an extension  $\psi$  is equivariant to the product action  $\psi_{\mu, \nu} \times \text{id}$ ;
- (3) If we are also given an  $S^1$ -action, denoted  $\phi_1$ , on  $S^1 \times D^2 \times \{1\}$  equivariant to  $\phi_{\mu, \mu-\nu}$  and compatible to  $\varphi$  on the boundary, then  $\varphi \cup \phi_1$  extends to a locally smooth  $S^1$ -action on  $S^1 \times D^2 \times I$ .

*Proof.* We first show (3). Let  $V_j$ ,  $j = 0, 1$ , be a subsolid torus of  $S^1 \times \text{int } D^2 \times \{j\}$  invariant under the action of  $\phi_j$ . Choose a closed domain  $W$  of  $S^1 \times \text{int } D^2 \times I$  homeomorphic to  $S^1 \times D^2 \times I$  such that  $W \cap (S^1 \times D^2 \times \{j\}) = V_j$ . Let  $\rho : W \rightarrow S^1 \times D^2 \times I$  be a homeomorphism, and  $V_t := \rho^{-1}(S^1 \times D^2 \times \{t\})$ ,  $t \in I$ . Let  $f_0 : (S^1 \times D^2, \phi_{\mu, \nu}) \rightarrow (V_0, \phi_0)$  and  $f_1 : (S^1 \times D^2, \phi_{\mu, \mu-\nu}) \rightarrow (V_1, \phi_1)$  be equivariant homeomorphisms. From the assumption,  $g_0 := f_1^{-1} \circ \rho_1^{-1} \circ \Pi_{0,1} \circ \rho_0 \circ f_0$  is isotopic the identity, where  $\rho_t := \rho|_{V_t} : V_t \rightarrow S^1 \times D^2 \times \{t\}$  and  $\Pi_{s,t} : S^1 \times D^2 \times \{s\} \rightarrow S^1 \times D^2 \times \{t\}$  is the natural identification. Let  $g_t$  be an isotopy between  $g_0$  and the identity ( $= g_1$ ). Then  $f_t := \rho_t^{-1} \circ \Pi_{1,t} \circ \rho_1 \circ f_1 \circ g_t$  gives a continuous family of homeomorphisms

$: S^1 \times D^2 \rightarrow V_t$  joining  $f_0$  and  $f_1$ . Using  $f_t$ , we can extend the orbit structure on  $V_0 \cup V_1$  to that on  $W$ . Thus we have an  $S^1$ -action  $\bar{\phi}$  on  $W$  extending  $(V_0, \phi_0)$  and  $(V_1, \phi_1)$ . Note  $S^1 \times D^2 \times I - \text{int } W \simeq S^1 \times A^2 \times I$ , where  $A^2$  is an annulus of  $D$  such that  $D - A^2$  is an open disk. From now on we identify  $S^1 \times D^2 \times I - \text{int } W$  with  $S^1 \times A^2 \times I$ . Now we have a free  $S^1$ -action  $\phi$  on  $S^1 \times \partial(A^2 \times I) \simeq T^3$  given by  $\varphi$ ,  $\phi_1$  and  $\bar{\phi}$ . Since  $\phi$  gives rise to a trivial bundle  $S^1 \rightarrow T^3 \rightarrow T^2$ , it provides an imbedding  $g : S^1 \times \partial(A^2 \times I) \rightarrow S^1 \times A^2 \times I$  such that

- (1)  $g$  leaves  $S^1 \times \partial A^2 \times \{t\}$  invariant for any  $t \in I$ ;
- (2)  $g(S^1 \times (x, t))$  gives the  $\phi$ -orbit for any  $(x, t) \in \partial(A^2 \times I)$ ;
- (3)  $g : \{1\} \times \partial(A^2 \times I) : \{1\} \times \partial(A^2 \times I) \rightarrow S^1 \times \partial(A^2 \times I)$  is a section to  $\phi$ .

Therefore we have a homeomorphism

$$h : S^1 \times \partial(A^2 \times I) - g(\{1\} \times \partial(A^2 \times I)) \simeq J \times \partial(A^2 \times I),$$

where  $J$  is an open interval. Extend the section  $g$  to an embedding  $G : \{1\} \times A^2 \times I \rightarrow S^1 \times A^2 \times I$  so that the  $\phi$ -orbits meet the  $G$ -image only with the  $g$ -image. Then  $h$  extends to a homeomorphism

$$H : S^1 \times A^2 \times I - G(\{1\} \times A^2 \times I) \simeq J \times A^2 \times I.$$

Since  $H$  extends to a homeomorphism  $\bar{H} : S^1 \times A^2 \times I \simeq S^1 \times A^2 \times I$  we can extend  $\phi$  to an  $S^1$ -action on  $S^1 \times A^2 \times I$ , proving (3).

(1) is immediate from (3) if one extend  $\varphi|_{S^1 \times \partial D^2 \times \{1\}}$  to an  $S^1$ -action  $\phi_1$  on  $S^1 \times D^2 \times \{1\}$  which is equivariant to  $\phi_{\mu, \mu - \nu}$ .

Note that a nontrivial Seifert bundle  $S^2 \times S^1 = \partial(S^1 \times D^2 \times I) \rightarrow S^2 = \partial(D^2 \times I)$  is essentially unique. It has two singular orbits, and the Seifert invariants are  $(\alpha, \beta)$  and  $(\alpha, \alpha - \beta)$  for some  $\alpha > \beta$  (see [40], p.459). Let  $\psi$  be any extension of  $\varphi$  to an  $S^1$ -action on  $S^1 \times D^2 \times I$ . From the above argument,  $E^*(\psi)$  consists of a segment along which Seifert invariants are  $(\mu, \nu)$ . Obviously  $F^*(\psi)$  is empty. Therefore Proposition 3.6 yields that  $\psi$  must be equivariant to  $\phi_{\mu, \nu} \times \text{id}$ .  $\square$

**Proposition 6.10.** *Suppose that  $B(p_i, r) \simeq S^1 \times D^3$  and  $m_i = 2$ . Then there exists a Seifert fibration  $\hat{f}_i : B^{f_i}(p_i, r) \rightarrow B(p, r)$  such that*

- (1)  $\hat{f}_i = f_i$  on  $A^{f_i}(p_i; r/100, r) - \bigcup_{j=1}^k A_j^{f_i}(p_i; \epsilon)$ ;
- (2)  $\hat{f}_i = \tilde{f}_i$  on  $\partial B^{f_i}(p_i, r)$ ;
- (3) the singular fibres of  $\hat{f}_i$  are contained in the union of two of  $\{A_j^i(p_i; \epsilon)\}$ , say,  $A_1^i(p_i; \epsilon) \cup A_2^i(p_i; \epsilon)$ ;
- (4) the singular locus  $\mathcal{C}_i$  of  $\hat{f}_i$  is a connected quasi-geodesic containing  $p$  in its interior and consisting of essential singular points of  $X^3$ ;
- (5) the Seifert invariants of the singular fibre of  $\hat{f}_i$  do not exceed

$$\frac{2\pi}{\max\{L(\Sigma_{\xi_1}(\Sigma_p)), L(\Sigma_{\xi_2}(\Sigma_p))\}},$$



where  $\xi_i$ ,  $i = 1, 2$ , are the directions at  $p$  represented by  $\mathcal{C}_i$

*Proof.* Identify  $A_j^{f_i}(p_i; \epsilon; r/100, r) \simeq (S^1 \times D^2) \times I$ . Now we have the  $S^1$ -action, say  $\phi_i$ , on  $A_j^{f_i}(p_i; \epsilon) \cap \partial B^{f_i}(p_i, r) = S^1 \times D^2 \times \{1\}$  given by Proposition 6.5. We also have the  $S^1$ -action on  $(S^1 \times \partial D^2) \times I$  coming from the  $S^1$ -bundle structure. Note that those two actions are compatible on the intersection  $S^1 \times \partial D^2 \times \{1\}$ . Using Lemma 6.9, we extend those  $S^1$ -actions to one on  $S^1 \times D^2 \times I$  which is equivariant to the product action of the action  $\phi_i$  on  $S^1 \times D^2 \times \{1\}$  and the trivial action on  $I$ . Thus we have extended the  $S^1$ -bundle  $\tilde{f}_i : A_j^{f_i}(p_i; r/100, r) - \bigcup_{j=1}^k A_j^{f_i}(p_i; \epsilon) \rightarrow A(p; r/100, r) - \bigcup_{j=1}^k A_j(p; \epsilon)$  to a Seifert bundle  $\bar{f}_i : A_j^{f_i}(p_i; r/100, r) \rightarrow A(p; r/100, r)$ . Next, extend it to a Seifert bundle  $\hat{f}_i : B^{f_i}(p_i, r) \rightarrow B(p, r)$  as follows. Since  $m_i = 2$ , the Seifert invariants of the Seifert bundle  $\partial B(p_i, r/100) \rightarrow \partial B(p, r/100)$ , the restriction of  $\bar{f}_i$ , are expressed as  $(\mu_i, \nu_i)$  and  $(\mu_i, \mu_i - \nu_i)$ . Therefore this Seifert bundle is isomorphic as a fibred space to the one on  $\partial(V_{\mu_i, \nu_i} \times I)$  induced from the product action  $\phi_{\mu, \nu} \times \text{id}$  on  $V_{\mu_i, \nu_i} \times I$ , where  $V_{\mu_i, \nu_i}$  denotes the fibred solid torus of type  $(\mu_i, \nu_i)$ . This provides a Seifert bundle  $B(p_i, r/100) \rightarrow B(p, r/100)$  such that

- (1) it is equivalent to the product  $V_{\mu_i, \nu_i} \times I$ ;
- (2) the Seifert bundle structures on  $A_j^{f_i}(p_i; r/100, r)$  and  $B(p_i, r/100)$  are compatible on their boundaries;

The properties (1), (2) and (3) are now obvious.

To prove (4) and (5), we first show

**Assertion 6.11.** *There exist  $r = r(p) > 0$  and  $\epsilon = \epsilon(p) > 0$  such that  $\partial B(p, s) \cap A_j(p; \epsilon)$  contains an essential singular point for each  $0 < s \leq r$  and  $j \in \{1, 2\}$ .*

*Proof.* Suppose that  $A(p; s_1, s_2) \cap A_j(p; \epsilon)$  does not meet  $ES(X^3)$  for some  $s_1 < s_2 \leq r$ . Then  $A_j^{f_i}(p_i; \epsilon; s_1, s_2)$  has an  $S^1$ -bundle structure compatible with  $f_i$ . This is a contradiction to Lemma 6.9.  $\square$

**Assertion 6.12.** *There exist  $r = r(p) > 0$  and  $\epsilon = \epsilon(p) > 0$  such that*

$$\frac{|d_p(y) - d_p(x)|}{d(x, y)} \geq 1 - \tau(r),$$

*for any  $x, y \in ES(X^3) \cap A_j(p; \epsilon)$  with  $y$  sufficiently close to  $x$  and for each  $j \in \{1, 2\}$ .*

*Proof.* Suppose the assertion does not hold. Then there would exist sequences  $x_i, y_i$  of  $ES(X^3) \cap A_j(p; \epsilon_i)$  with  $r = r_i \rightarrow 0$  and  $\epsilon_i \rightarrow 0$  such that

$$(1) \quad \frac{|d_p(y_i) - d_p(x_i)|}{d(x_i, y_i)} \leq 1 - \mu,$$

for some  $\mu > 0$ ;

(2)  $s_i := d(p, x_i) \rightarrow 0$  and  $d(x_i, y_i)/s_i \rightarrow 0$ .

We may assume that  $(\frac{1}{r_i}X^3, x_i)$  converges to a space  $(Y^3, x_\infty)$ , where  $Y^3$  is isometric to the tangent cone  $K_p$ . Putting  $\epsilon_i = d(x_i, y_i)$ , we consider any limit  $(\hat{Y}^3, \hat{x}_\infty)$  of another rescaling  $(\frac{1}{\epsilon_i}X^3, x_i)$ , where  $\hat{Y}^3$  is isometric to a product  $\mathbb{R} \times Y_0$ . Let  $\hat{y}_\infty$  denote the limit of  $y_i$  under this convergence. Since  $\hat{x}_\infty, \hat{y}_\infty \in ES(\hat{Y}^3)$ , and since  $\pi(\hat{x}_\infty) \neq \pi(\hat{y}_\infty)$ , where  $\pi : \hat{Y}^3 \rightarrow Y_0$  is the projection,  $Y_0$  contains two essential singular points with distance, say  $a < 1$ . Thus  $Y_0$  must be isometric to the double  $D([0, a] \times [0, \infty))$ . In particular,  $\dim \hat{Y}(\infty) = 1$  while  $\dim Y(\infty) = 2$ . This contradicts Lemma 4.4.  $\square$

From Assertions 6.11 and 6.12,  $ES(X^3) \cap (A_1(p; \epsilon) \cup A_2(p; \epsilon))$  defines a continuous curve, denoted  $\mathcal{C}_p$ , through  $p$ , such that  $\Sigma_x(\mathcal{C}_p)$  contains at least two elements for any  $x \in \mathcal{C}_p$ . Therefore  $\mathcal{C}_p$  is a quasigeodesic by Proposition 5.7. Obviously we can arrange the fibre structure defined by  $\hat{f}_i$  so that the singular locus  $\mathcal{C}_i$  of  $\hat{f}_i$  coincides with  $\mathcal{C}_p$ . (5) follows from Proposition 6.5. This completes the proof of Proposition 6.10.  $\square$

Now we consider the other case.

*Case II.*  $B(p_i, r) \simeq D^4$ .

In this case, by [32], the Seifert bundle  $\tilde{f}_i : \partial B^{f_i}(p_i, r) \simeq S^3 \rightarrow \partial B(p, r) \simeq S^2$  is actually given by an  $S^1$ -action equivariant to the restriction of a canonical action  $\psi_{a_i b_i}$  on  $D^4(1)$  to  $S^3(1) = \partial D^4(1)$  for suitable relatively prime integers  $a_i, b_i$ . From Lemma 3.5, the Seifert invariants of  $\tilde{f}_i : S^3 \rightarrow S^2$  have forms  $(a_i, a'_i)$  and  $(b_i, b'_i)$  with

$$\begin{vmatrix} a_i & b_i \\ a'_i & b'_i \end{vmatrix} = \pm 1.$$

**Proposition 6.13.** *Suppose  $B(p_i, r) \simeq D^4$ . Then*

- (1)  $p$  is an extremal point of  $X^3$ ;
- (2) there is an  $i_0$  such that for every  $i \geq i_0$  we have a locally smooth  $S^1$ -action  $\psi_i$  on  $B^{f_i}(p_i, r)$  extending  $\tilde{f}_i$  such that the Seifert bundle  $\hat{f}_i : A^{f_i}(p_i; r/100, r) \rightarrow A(p; r/100, r)$  given by  $\psi_i$  satisfies the following:
  - (a)  $\hat{f}_i = f_i$  on  $A^{f_i}(p_i; r/100, r) - \bigcup_{j=1}^k A_j^{f_i}(p_i; \epsilon; r/100, r)$ ;
  - (b) the singular locus of  $\hat{f}_i$  (if it exists) extends to a continuous quasi-geodesic  $\mathcal{C}_i$  containing  $p$ .  $\mathcal{C}_i$  is contained in the union of at most two of  $A_j(p; \epsilon)$ 's, and consisting of essential singular points of  $X^3$ ;
  - (c)  $\psi_i$  is equivariant to the canonical action  $\psi_{a_i, b_i}$  for some relatively prime integers  $a_i$  and  $b_i$ , where

$$|a_i| \leq \frac{2\pi}{L(\Sigma_{\xi_1}(\Sigma_p))}, \quad |b_i| \leq \frac{2\pi}{L(\Sigma_{\xi_2}(\Sigma_p))},$$

for some  $\xi_1 \neq \xi_2$  and at least one of them is a direction of  $\mathcal{C}_i$  at  $p$ , and the other is another direction of  $\mathcal{C}_i$  if it exists.

*Proof.* (1) follows from Proposition 6.3. (2)-(a), (b) are similar to Proposition 6.10. (2)-(c) follows from an argument similar to Proposition 6.5.  $\square$

## 7. COLLAPSING TO THREE-SPACES WITHOUT BOUNDARY- GLOBAL CONSTRUCTION

Let a sequence of closed orientable 4-dimensional Riemannian manifolds  $M_i^4$  with  $K \geq -1$  converge to a 3-dimensional compact Alexandrov space  $X^3$  without boundary. By Fibration Theorem 1.2, we have a locally trivial  $S^1$ -bundle  $f_i : M_i' \rightarrow X^3 - S_{\delta_3}(X^3)$  which is an  $\epsilon_i$ -approximation,  $\lim_{i \rightarrow \infty} \epsilon_i = 0$ , where  $M_i'$  is an open subset of  $M_i^4$ . The purpose of this section is to construct a globally defined locally smooth, local  $S^1$ -action on  $M_i^4$ .

**Theorem 7.1.** *Suppose that  $M_i^4$  collapses to a 3-dimensional compact Alexandrov space  $X^3$  without boundary under  $K \geq -1$ . Then there exists a locally smooth, local  $S^1$ -action  $\psi_i$  on  $M_i^4$  satisfying the following:*

- (1)  $M_i^4/\psi_i \simeq X^3$ ;
- (2) We have an inclusion  $F^*(\psi_i) \subset \text{Ext}(X^3)$ ;
- (3) For each  $x_\alpha \in F^*(\psi_i)$ , there are  $p_i^\alpha \in M_i^4$  close to  $x_\alpha$  and  $r_\alpha > 0$  independent of  $i$  such that  $B^{f_i}(p_i^\alpha, r_\alpha)$  is  $\psi_i$ -invariant and  $(B^{f_i}(p_i^\alpha, r_\alpha), \psi_i)$  is equivariantly homeomorphic to  $(D^4(1), \psi_{a_i b_i})$  for some relatively prime integers  $a_i, b_i$  satisfying

$$|a_i| \leq \frac{2\pi}{\ell_1}, \quad |b_i| \leq \frac{2\pi}{\ell_2},$$

where  $\ell_1$  and  $\ell_2$  are described as in Proposition 6.13 (2);

- (4) Each connected component  $\mathcal{C}_i \subset X^3$  of the singular locus  $S^*(\psi_i)$  is either a periodic quasi-geodesic in  $X^3 - F^*(\psi_i)$  or a quasi-geodesic path or loop in  $X^3$  connecting some points of  $F^*(\psi_i)$ ;
- (5) The order of the isotropy subgroup along every component  $\hat{\mathcal{C}}_i$  of the exceptional locus  $E^*(\psi)$  does not exceed

$$\inf_{x \in \hat{\mathcal{C}}_i} \frac{2\pi}{L_{\xi_x}(\Sigma_x)},$$

where  $\xi_x$  is a direction at  $x$  defined by  $\hat{\mathcal{C}}_i$ .

**Remark 7.2.** Theorem 7.1 (2) means that for any fixed point  $p_i^\alpha \in \text{Fix}(\psi_i)$ , there corresponds a unique extremal point  $x_\alpha \in \text{Ext}(X^3)$  which is Gromov-Hausdorff close to  $p_i^\alpha$ .

**Example 7.3.** For relatively prime integers  $a, b$ , let us consider the restriction of the canonical action  $\psi_{ab}$  on  $D^4(1)$  to  $S^3(1)$ . This naturally extends to an isometric  $S^1$ -action  $\tilde{\psi}_{a,b}$  on  $S^4(1)$ , the spherical

suspension over  $S^3(1)$ , with two fixed points, say  $x_1$  and  $x_2$ . Let  $X^3$  be the quotient space  $S^4(1)/\tilde{\psi}_{a,b}$ , which is the spherical suspension over  $S^3(1)/\psi_{a,b} \simeq S^2$ . By [46], we have a sequence of metrics  $g_i$  on  $S^4$  such that

- (1) the sectional curvature  $K_{g_i}$  has a uniform lower bound;
- (2)  $(S^4, g_i)$  collapses to  $X^3$ .

Note that

- (3) the diameters of the spaces (say  $\Sigma$ ) of directions at the poles of  $X^3$  are both equal to  $\pi/2$ ;
- (4)  $\Sigma$  has at most two singular points, say  $\xi, \eta$ , realizing the diameter of it, where

$$L(\Sigma_\xi(\Sigma)) = 2\pi/|a|, \quad L(\Sigma_\eta(\Sigma)) = 2\pi/|b|.$$

The singular locus  $S^*(\tilde{\psi}_{a,b})$  of  $\tilde{\psi}_{a,b}$  is a simple closed loop (resp. a simple arc) joining the two poles if  $|a| \geq 2$  and  $|b| \geq 2$  (resp. if just one of  $|a|$  and  $|b|$  is greater than 1).

**Example 7.4.** Consider the  $S^1$ -action  $\psi$  on  $\mathbb{C}P^2$  given by

$$z \cdot [z_0 : z_1 : z_2] = [z^{a_0} z_0 : z^{a_1} z_1 : z^{a_2} z_2],$$

for suitably chosen integers  $a_i$ ,  $0 \leq i \leq 2$ . We assume that  $a_{i+1} - a_i$  and  $a_{i+2} - a_i$  are relatively prime, where  $i$  is understood in *mod* 3. Note that  $F^*(\psi)$  consists of  $x_0 = [1 : 0 : 0]$ ,  $x_1 = [0 : 1 : 0]$ , and  $x_2 = [0 : 0 : 1]$ . The action  $\psi$  induces actions equivariant to the canonical actions of type  $(a_{i+1} - a_i, a_{i+2} - a_i)$  on a neighborhood of  $x_i$ . For an invariant metric  $g$  on  $\mathbb{C}P^2$ , take a sequence of metrics  $g_i$  on  $\mathbb{C}P^2$  such that

- (1) the sectional curvature  $K_{g_i}$  has a uniform lower bound;
- (2)  $(\mathbb{C}P^2, g_i)$  converges to the quotient space  $X^3 = (\mathbb{C}P^2, g)/S^1$ .

Note that

- (3)  $\text{diam}(\Sigma_{\bar{x}_i}) = \pi/2$ , where  $\bar{x}_i \in X^3$  is the image of  $x_i$  under the projection  $\mathbb{C}P^2 \rightarrow X^3$ ;
- (4)  $\Sigma_{\bar{x}_i}$  has exactly two singular points, say  $\xi_{i,i+1}, \xi_{i,i+2}$  realizing the diameter of  $\Sigma_{\bar{x}_i}$ , where

$$L(\Sigma_{\xi_{i,i+1}}(\Sigma_{\bar{x}_i})) = 2\pi/|a_{i+1} - a_i|, \quad L(\Sigma_{\xi_{i,i+2}}(\Sigma_{\bar{x}_i})) = 2\pi/|a_{i+2} - a_i|.$$

Now suppose  $a_0 = 0$  for instance. Then the singular locus  $S^*(\psi)$  is a simple arc joining  $x_1$  and  $x_2$  through  $x_0$ . If  $a_{i+1} - a_i$  and  $a_{i+2} - a_i$  are relatively prime for every  $i$  in *mod* 3, then  $\mathcal{C}$  is a simple closed loop joining  $x_0, x_1$  and  $x_2$ .

**Example 7.5.** Let  $S^2(1) \subset \mathbb{C} \times \mathbb{R}$  be the unit sphere and consider the  $S^1$ -action  $\psi$  on  $S^2(1) \times S^2(1)$  given by

$$z \cdot (z_1, t_1, z_2, t_2) = (z^a z_1, t_1, z^b z_2, t_2),$$

for relatively prime integers  $a$  and  $b$ , where  $z_i \in \mathbb{C}$ ,  $t_i \in \mathbb{R}$ ,  $i = 1, 2$ . The action  $\psi$  induces actions equivariant to the canonical actions of

type  $(a, b)$  on a neighborhood of each of the four fixed point of  $\psi$ , up to orientation. We can take a sequence of metrics  $g_i$  on  $S^2(1) \times S^2(1)$  such that

- (1) the sectional curvature  $K_{g_i}$  has a uniform lower bound;
- (2)  $(S^2(1) \times S^2(1), g_i)$  converges to the quotient space  $X^3 = (S^2(1) \times S^2(1))/S^1$ .

Note that  $X^3$  is an Alexandrov space with nonnegative curvature having no boundary. The singular locus  $S^*(\psi)$  consists of (i) a simple loop joining those four fixed points if  $|a|, |b| \neq 1$ , (ii) two segments joining those four fixed points if just one of  $|a|$  and  $|b|$  is equal to 1, (iii) those four fixed points if both  $|a|$  and  $|b|$  are equal to 1.

In view of Examples 7.3, 7.4 and 7.5, taking connected sum around fixed points, we can construct an  $S^1$ -action on every connected sum of  $S^4$ ,  $\pm \mathbb{C}P^2$  and  $S^2 \times S^2$  whose orbit space  $X^3$  has no boundary.

*Proof of Theorem 7.1.* Recall that for each  $p \in S_{\delta_3}(X^3)$ , there are sufficiently small  $\epsilon = \epsilon_p > 0$  and  $r = r(p, \epsilon) > 0$  for which we have a locally smooth  $S^1$ -action  $\psi_{p,i}$  on  $B^{f_i}(p_i, r)$  for sufficiently large  $i$ , where  $p_i \in M_i^4$  is close to  $p$ . To prove Theorem 7.1, we patch those  $S^1$ -actions and a local  $S^1$ -action defined by  $f_i$  to obtain a globally defined locally smooth, local  $S^1$ -action. Note that if  $p$  is an isolated point of  $S_{\delta_3}(X^3)$ , we have nothing to do for the patching.

For any non-isolated point  $p$  of  $S_{\delta_3}(X^3)$ , let  $\epsilon = \epsilon_p$  and  $r = r(p, \epsilon)$  be so small that

- (1)  $\{A_j(p; \epsilon)\}_{j=1, \dots, k(p)}$  covers  $B(p, 2r) \cap S_{\delta_3}(X^3)$ , where  $A_j(p; \epsilon) = B(px^j, \epsilon r) \cap B(p, 2r)$  is defined as in the previous section;
- (2)  $(A_j(p; \epsilon) - A_j(p; \epsilon^{10})) \cap A(p; r/100, 2r)$  does not meet  $S_{\delta_3}(X^3)$ ;
- (3)  $\{A_j(p; \epsilon) - B(p, \epsilon/100)\}_j$  is disjoint.

From the compactness of  $S_{\delta_3}(X^3)$ , take  $p^1, \dots, p^N$  such that  $S_{\delta_3}(X^3) \subset \bigcup_{\alpha=1}^N B(p^\alpha, r_\alpha/3)$ , where  $r_\alpha = r(p^\alpha, \epsilon_\alpha)$  and  $\epsilon_\alpha$  are defined as above. We assume that  $r_1 \geq r_2 \geq \dots \geq r_N$ . Note that if  $d(p^\alpha, p^\beta) \leq r_\alpha/3$  with  $\alpha < \beta$ , then  $B(p^\alpha, r_\alpha) \supset B(p^\beta, r_\beta/3)$ . Hence considering the covering  $\{B(p^\alpha, r_\alpha)\}$  of  $S_{\delta_3}(X^3)$  in stead of  $\{B(p^\alpha, r_\alpha/3)\}$ , and removing some of  $\{r_2, \dots, r_N\}$  if necessary, we may assume that  $B(p^\alpha, r_\alpha/3)$  does not contain  $p^\beta$  for every  $1 \leq \alpha < \beta \leq N - 1$ . In particular,  $\{B(p^\alpha, r_\alpha/6)\}$  is disjoint.

Let  $p_i^\alpha \in M_i^4$  be a point close to  $p^\alpha$ . By Fibration Theorem 1.2, we have an  $S^1$ -bundle structure  $\mathcal{F}_i$  on  $f_i^{-1}(X^3 - S_{\delta_3}(X^3)) \subset M_i^4$ . On each  $B^{f_i}(p_i^\alpha, r_\alpha)$ , we have the structure  $\mathcal{S}_i^\alpha$  of locally smooth  $S^1$ -action  $\psi_{p_i^\alpha, i}$  given by Proposition 6.10 or 6.13. We have to patch all of those structures  $\mathcal{F}_i$  and  $\mathcal{S}_i^\alpha$  together to obtain a globally defined structure on  $M_i^4$  of locally smooth, local  $S^1$ -action as stated in Theorem 7.1.

Let  $A_{j_\alpha}(p^\alpha; \epsilon_\alpha)$ ,  $A_{j_\alpha}(p^\alpha; \epsilon_1; r_1), \dots, 1 \leq j_\alpha \leq k(p^\alpha)$ , be as defined right after Lemma 6.7 for  $p^\alpha$ . Now we are going to patch  $\mathcal{F}_i$ ,  $\mathcal{S}_i^\alpha$  and  $\mathcal{S}_i^\beta$  assuming  $A_{j_\alpha}(p^\alpha; \epsilon_\alpha; r_\alpha) \cap A_{j_\beta}(p^\beta; \epsilon_\beta; r_\beta)$  to be nonempty with  $\alpha < \beta$ .

If  $A_{j_\alpha}(p^\alpha; \epsilon_\alpha; r_\alpha) \cap A_{j_\beta}(p^\beta; \epsilon_\beta; r_\beta)$  does not meet  $S_{\delta_3}(X^3)$ , there is no need to patch  $\psi_{p^\alpha, i}$  and  $\psi_{p^\beta, i}$  on the intersection  $B^{f_i}(p_i^\alpha, r_\alpha) \cap B^{f_i}(p_i^\beta, r_\beta)$ . Hence assume that there is a point  $x$  in the intersection of  $A_{j_\alpha}(p^\alpha; \epsilon_\alpha; r_\alpha)$ ,  $A_{j_\beta}(p^\beta; \epsilon_\beta; r_\beta)$  and  $S_{\delta_3}(X^3)$ .

We consider the following two cases depending on the value  $\angle xp^\alpha p^\beta$ .

*Case I.*  $\angle xp^\alpha p^\beta > \epsilon_\alpha/2$ .

Then  $p^\beta$  must be contained in some  $A_{k_\alpha}(p^\alpha; \epsilon_\alpha; r_\alpha)$  with  $k_\alpha \neq j_\alpha$ . Extending the  $S^1$ -action on  $B^{f_i}(p^\alpha, r_\alpha)$  to one on  $B^{f_i}(p^\alpha, 2r_\alpha)$  in a similar way, we can borrow the  $S^1$ -action structure on  $B^{f_i}(p^\alpha, 2r_\alpha)$  as a compatible  $S^1$ -action structure on  $A_{j_\beta}(p^\beta; \epsilon_\beta; r_\beta)$ . See Figure 2

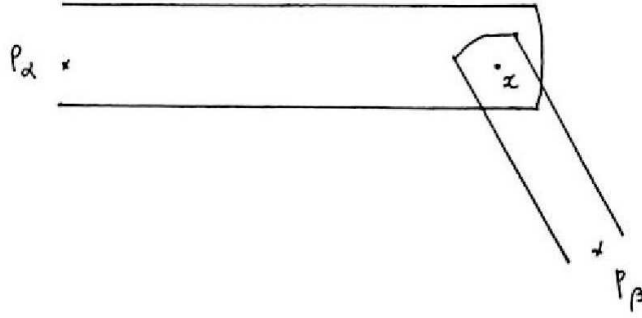


FIGURE 2.

*Case II.*  $\angle xp^\alpha p^\beta \leq \epsilon_\alpha/2$ .

In this case, we actually have  $p^\beta \in A_{j_\alpha}(p^\alpha; \epsilon_\alpha^{10}; r_\alpha)$  if  $d(x, p^\beta) \leq \epsilon_\alpha r_\alpha$ , then taking  $r_\alpha$  slightly larger if necessary, we may assume that  $B^{f_i}(p_i^\beta, r_\beta) \subset B^{f_i}(p_i^\alpha, r_\alpha)$ . Therefore we can remove  $\beta$  from  $\{\alpha\}$ , and we may assume  $d(x, p^\beta) > \epsilon_\alpha r_\alpha$ . It follows that  $\angle p^\alpha x p^\beta$  is sufficiently small or sufficiently close to  $\pi$ . In the former case, taking  $r_\alpha$  slightly larger if necessary, we may assume that  $B^{f_i}(p_i^\beta, r_\beta) \subset B^{f_i}(p_i^\alpha, r_\alpha)$ . Therefore we can remove  $\beta$  from  $\{\alpha\}$  as before. In the latter case, assuming  $\epsilon_\alpha r_\alpha \geq \epsilon_\beta r_\beta$  (the other case is proved similarly), we have

$$A_{j_\beta}^{f_i}(p_i^\beta; \epsilon_\beta; r_\beta/100, r_\beta) \cap \text{int } B^{f_i}(p_i^\alpha, r_\alpha) \subset \text{int } A_{j_\alpha}^{f_i}(p_i^\alpha; 2\epsilon_\alpha; r_\alpha/100, r_\alpha).$$

See Figure 3.

We perturb  $\mathcal{S}_i^\alpha$  to a new locally smooth  $S^1$ -action  $\tilde{\mathcal{S}}_i^\alpha$  on  $A_{j_\alpha}^{f_i}(p_i^\alpha; 2\epsilon_\alpha; r_\alpha/100, r_\alpha) \cap B^{f_i}(p_i^\beta, r_\beta)$  as follows: we let  $\tilde{\mathcal{S}}_i^\alpha$  coincide

- with  $\mathcal{S}_i^\beta$  on  $A_{j_\alpha}^{f_i}(p_i^\alpha; 2\epsilon_\alpha; r_\alpha/100, r_\alpha) \cap A_{j_\beta}^{f_i}(p_i^\beta; \epsilon_\beta; r_\beta/100, r_\beta)$  and
- ;

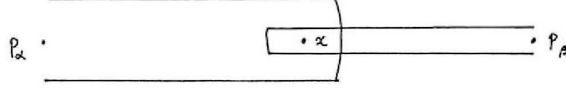


FIGURE 3.

- with  $\mathcal{F}_i$  on the complement of  $A_{j_\beta}^{f_i}(p_i^\beta; \epsilon_\beta; r_\beta/100, r_\beta)$  in  $A_{j_\alpha}^{f_i}(p_i^\alpha; 2\epsilon_\alpha; r_\alpha/100, r_\alpha) \cap B^{f_i}(p_i^\beta, r_\beta)$ .

Note that this is well-defined. Using

$$A_{j_\alpha}^{f_i}(p_i^\alpha; 2\epsilon_\alpha; r_\alpha/100, r_\alpha) - \text{int } B^{f_i}(p^\beta, r_\beta) \simeq (S^1 \times D^2) \times I,$$

together with lemma 6.9, we can extend  $\tilde{\mathcal{S}}_i^\alpha$  to a locally smooth  $S^1$ -action on the complement of  $B^{f_i}(p_i^\beta, r_\beta)$  in  $A_{j_\alpha}^{f_i}(p_i^\alpha; \epsilon_\alpha; r_\alpha/100, r_\alpha)$ , which is compatible with  $\mathcal{S}_i^\alpha$  on  $\{d_{p_i^\alpha} = r_\alpha/100\} \cap A_{j_\alpha}^{f_i}(p_i^\alpha; \epsilon_\alpha; r_\alpha/100, r_\alpha)$ . Thus we obtain the required locally smooth  $S^1$ -action on the union of  $A_{j_\alpha}^{f_i}(p_i^\alpha; 2\epsilon_\alpha; r_\alpha/100, r_\alpha)$  and  $A_{j_\beta}^{f_i}(p_i^\beta; \epsilon_\beta; r_\beta/100, r_\beta)$ , which is compatible with  $\tilde{\mathcal{S}}_i^\alpha$ ,  $\mathcal{F}_i$  and  $\mathcal{S}_i^\beta$ . Repeating this local patching procedure finitely many times, we obtain the global collapsing structure and complete the proof of Theorem 7.1.  $\square$

## 8. COLLAPSING TO THREE-SPACES WITH BOUNDARY

In this section, we consider the situation that a sequence of closed orientable 4-dimensional Riemannian manifolds  $M_i^4$  with  $K \geq -1$  converge to a 3-dimensional compact Alexandrov space  $X^3$  with boundary. We construct a globally defined locally smooth, local  $S^1$ -action on  $M_i^4$  to complete the proof of Theorem 0.2.

**Theorem 8.1.** *Suppose that  $M_i^4$  collapses to a 3-dimensional compact Alexandrov space  $X^3$  with boundary under  $K \geq -1$ . Then there exists a locally smooth, local  $S^1$ -action  $\psi_i$  on  $M_i^4$  satisfying the following:*

- (1)  $M_i^4/\psi_i \simeq X^3$ ;
- (2)  $F^*(\psi_i)$  coincides with the union of  $\partial X^3$  and a subset of  $\text{Ext}(\text{int } X^3)$ , say  $\{x_\alpha\}_{\alpha=1, \dots, k}$ ;
- (3) For each  $1 \leq \alpha \leq k$ , there is an  $r_\alpha > 0$  independent of  $i$  such that if  $p_i^\alpha \in M_i^4$  denotes the  $\psi_i$ -fixed point corresponding to  $x_\alpha$ , then  $B^{f_i}(p_i^\alpha, r_\alpha)$  is  $\psi_i$ -invariant and  $(B^{f_i}(p_i^\alpha, r_\alpha), \psi_i)$  is equivariantly homeomorphic to  $(D^4(1), \psi_{a_i, b_i})$  for some relatively prime integers  $a_i, b_i$  satisfying

$$|a_i| \leq \frac{2\pi}{\ell_1}, \quad |b_i| \leq \frac{2\pi}{\ell_2},$$

where  $\ell_1$  and  $\ell_2$  are described as in Proposition 6.13;

- (4) Each connected component  $\mathcal{C}_i$  of the singular locus of  $\psi_i$  in  $\text{int } X^3$  is either a periodic quasi-geodesic in  $\text{int } X^3 - \{x_\alpha\}$  or a quasi-geodesic path or loop in  $\text{int } X^3$  connecting some of  $\{x_\alpha\}$ ;
- (5) The order of the isotropy subgroup along every component  $\hat{\mathcal{C}}_i$  of the exceptional locus  $E^*(\psi)$  does not exceed

$$\inf_{x \in \hat{\mathcal{C}}_i} \frac{2\pi}{L(\Sigma_{\xi_x}(\Sigma_x))},$$

where  $\xi_x$  is a direction at  $x$  defined by  $\hat{\mathcal{C}}_i$ .

**Example 8.2.** Let us consider the  $S^1$ -action  $\psi$  on  $S^4 = D^3 \times S^1 \cup S^2 \times D^2$ , where  $S^1$  acts canonically only on  $S^1$ -factors and  $D^2$ -factors of  $D^3 \times S^1$  and  $S^2 \times D^2$  respectively. Then  $S^4/\psi = D^3$  and  $\text{Fix}(\psi) = \partial D^3$ . Fixing a  $\psi$ -invariant metric  $g$  on  $S^4$ , by [46], we have a sequence of metrics  $g_i$  on  $S^4$  such that  $(S^4, g_i)$  collapses to  $(S^4, g)/\psi$  under a lower sectional curvature bound.

**Example 8.3.** Consider the  $D^2$ -bundle  $S^2 \tilde{\times}_{-1} D^2$  over  $S^2$  with the Euler number  $-1$  and choose a fibre metric on  $S^2 \tilde{\times}_{-1} D^2$  whose fibre is isometric to the unit disk  $D^2(1)$ . Let us consider the  $S^1$ -action  $\psi$  on  $\mathbb{CP}^2 = D^4(1) \cup_{S^3} S^2 \tilde{\times}_{-1} D^2$  such that the action of  $\psi$  coincides with the canonical action  $\psi_{1,1}$  on  $D^4(1)$  and  $\psi$  acts on each  $S^2 \tilde{\times}_{-1} S^1(r) \subset S^2 \tilde{\times}_{-1} D^2$ ,  $0 < r \leq 1$ , as the Hopf fibration. Then  $F(\psi)$  is the disjoint union of the origin of  $D^4(1)$  and the zero-section of  $S^2 \tilde{\times}_{-1} D^2$ , and  $\mathbb{CP}^2/\psi \simeq D^3$ . For an invariant metric  $g$  on  $\mathbb{CP}^2$ , we obtain a sequence of metrics  $g_i$  on  $\mathbb{CP}^2$  such that  $(\mathbb{CP}^2, g_i)$  collapses to the quotient space  $X^3 = (\mathbb{CP}^2, g)/\psi$  under a lower sectional curvature bound.

**Example 8.4.** Let us consider the  $S^1$ -action on  $S^2 \times S^2$  such that  $S^1$  acts only on one  $S^2$ -factor. Then  $S^2 \times S^2/S^1 = I \times S^2$  and the fixed point set is the disjoint union of two copies of  $S^2$ . For an invariant metric  $g$  on  $S^2 \times S^2$ , we obtain a sequence of metrics  $g_i$  on  $S^2 \times S^2$  such that the sectional curvature  $K_{g_i}$  has a uniform lower bound and  $(S^2 \times S^2, g_i)$  collapses to the quotient space  $X^3 = (S^2 \times S^2, g)/S^1$ .

In Section 7, we have constructed  $S^1$ -action on every connected sum of  $S^4$ ,  $\pm \mathbb{CP}^2$  and  $S^2 \times S^2$  whose orbit space  $X^3$  has no boundary. In view of Examples 8.2, 8.3 and 8.4 together with the construction in the previous section, taking connected sum around fixed points, we can construct an  $S^1$ -action on every connected sum of  $S^4$ ,  $\pm \mathbb{CP}^2$  and  $S^2 \times S^2$  whose orbit space  $X^3$  has nonempty boundary.

Let us go back to the situation of Theorem 8.1. Take a small  $\nu > 0$  such that  $X^3 - X_\nu^3$  provides a collar neighborhood of  $\partial X^3$  (Theorem 5.14). By Theorem 7.1, we obtain a closed domain  $M_{\nu,i}^4$  of  $M_i^4$  such that

- (1) it collapses to  $X_\nu^3$ ;



- (2) there exists a locally smooth, local  $S^1$ -action on  $M_{\nu i}^4$  such that  $M_{\nu i}^4/S^1 \simeq X_\nu^3$ .

Let  $M_i^\partial$  denote the closure of  $M_i^4 - M_{\nu i}^4$ . The rest of this section is almost devoted to prove

**Theorem 8.5.**  $M_i^\partial$  is homeomorphic to a  $D^2$ -bundle over  $\partial X_\nu^3$  compatible with the  $S^1$ -bundle structure of  $\partial M_{\nu i}^4$  induced from the local  $S^1$ -action on  $M_{\nu i}^4$ .

From Theorem 8.5,  $M_i^4$  is a gluing of  $M_{\nu i}^4$  equipped with the local  $S^1$ -action and a  $D^2$ -bundle over  $\partial X_\nu^3$ , where the  $S^1$ -orbit over a point  $x \in \partial X_\nu^3$  is identified with the boundary of the  $D^2$ -fibre over  $x$ . Therefore it is straightforward to construct a desired globally defined, locally smooth, local  $S^1$ -action on  $M_i^4$  satisfying the conclusions of Theorem 8.1.

A key in the proof of Theorem 8.5 is the following lemma.

- Lemma 8.6.** (1) For a point  $p \in \partial X^3$ , take  $p_i \in M_i^4$  with  $p_i \rightarrow p$ . Then there exists a positive number  $r_0$  such that  $B(p_i, r)$  is homeomorphic to  $D^4$  for every  $r \leq r_0$  and each sufficiently large  $i \geq i_0$ , where  $i_0 = i_0(r)$ ;  
(2) There are no singular orbits over  $\partial X_\nu^3$ , namely  $\partial M_{\nu i}^4$  is an  $S^1$ -bundle over  $\partial X_\nu^3$ .

By Theorem 4.12,

$$(8.1) \quad B(p_i, r) \simeq D^4 \quad \text{or} \quad S^1 \times D^3.$$

We have to exclude the second possibility in (8.1).

Choose a  $\delta_1 r$ -net  $N(\delta_1)$  of  $A(p; r/10, 2r) \cap \partial X$  and take a finite subset  $N_i(\delta_1) \subset M_i^4$  converging to  $N(\delta_1)$ , and suppose that  $d_{GH}(M_i^4, X^3) < \epsilon_i$  and  $\epsilon_i \ll \delta_1 r$ .

Consider a smooth approximation

$$f_i = \tilde{d}(N_i(\delta_1), \cdot).$$

Since  $f := d(N(\delta_1), \cdot)$  is regular on  $f^{-1}([\delta_2 r/100, \delta_2 r])$ , where  $\delta_1 \ll \delta_2 \ll 1$ ,  $f_i$  is also regular on  $f_i^{-1}([\delta_2 r/100, \delta_2 r])$ .

**Sublemma 8.7.** There exist positive numbers  $\nu$  and  $r$  with  $\nu \ll r$  such that for any sufficiently large  $i \geq i(\nu, r)$  we have a closed domain  $M_i(p_i, r, \nu)$  satisfying

- (1)  $M_i(p_i, r, \nu)$  converges to  $A(p; r/10, 2r) \cap X_\nu^3$  under the convergence  $M_i^4 \rightarrow X^3$ ;
- (2)  $M_i(p_i, r, \nu) \cap \partial B(p_i, r)$  is homeomorphic to a Seifert bundle over  $\partial B(p, r) \cap X_\nu^3 \simeq D^2$  having at most one singular orbit;
- (3)  $M_i(p_i, r, \nu)$  is homeomorphic to  $(M_i(p_i, r, \nu) \cap \partial B(p_i, r)) \times I$ .

*Proof.* Take  $\nu > 0$  such that  $\text{int } \Sigma_p - (\Sigma_p)_\nu$  does not meet  $S_{\delta_3}(\Sigma_p)$ . Then in view of the convergence  $(\frac{1}{r}X^3, p) \rightarrow (K_p, o_p)$  as  $r \rightarrow 0$ , we

have such an  $r_0 > 0$  that  $A(p; r/10, 2r) \cap (X_{\nu_{10r}}^3 - X_{\nu_r}^3)$  does not meet  $S_{\delta_4}(X^3)$  for any  $r \leq r_0$ . Note that  $d_{p, GH}((\frac{1}{r}M_i^4, p_i), (K_p, o_p)) < \epsilon'_r + \epsilon_i/r$  with  $\lim_{r \rightarrow 0} \epsilon'_r = 0$ ,  $\lim_{i \rightarrow \infty} \epsilon_i = 0$ , and that  $\Sigma_p$  satisfies the assumption of Lemma 5.4. Hence if  $r$  is sufficiently small and  $i$  is sufficiently large  $i \geq i(r)$ , Theorem 7.1 applied to  $A(o_p; 1/10, 2) \cap (K_p)_\nu$  gives a closed domain  $M_i(p_i, r, \nu)$  satisfying (1) and (2).

Using  $f_i$ -flow curves, we can slightly deform  $M_i(p_i, r, \nu)$  to a closed domain  $M'_i(p_i, r, \nu)$  in such a way that it is  $\tilde{d}_{p_i}$ -flow-invariant. Therefore  $M'_i(p_i, r, \nu)$  is homeomorphic to  $(M'_i(p_i, r, \nu) \cap \partial B(p_i, r)) \times I$ , yielding (3).  $\square$

Putting  $M_i^\partial(p_i, r, \nu)$  to be the closure of  $A(p_i; r/10, 2r) - M_i(p_i, r, \nu)$ , we next investigate the topology of  $M_i^\partial(p_i, r, \nu) \cap \partial B(p_i, r)$ .

In a way similar to Sublemma 8.7 (3), we obtain

$$M_i^\partial(p_i, r, \nu) \cap A(p_i; r/5, r) \simeq (M_i^\partial(p_i, r, \nu) \cap \partial B(p_i, r)) \times I.$$

**Sublemma 8.8.**  $M_i^\partial(p_i, r, \nu) \cap \partial B(p_i, r)$  is a  $D^2$ -bundle over  $\partial X_\nu \cap \partial B(p, r) \simeq S^1$  which is compatible with the  $S^1$ -bundle over  $\partial X_\nu \cap \partial B(p, r)$  induced from the fibre structure on  $M_i(p_i, r, \nu) \cap \partial B(p_i, r)$ .

In particular  $M_i^\partial(p_i, r, \nu) \cap \partial B(p_i, r)$  is homeomorphic to  $D^2 \times S^1$ .

*Proof.* For a small  $\epsilon_1 > 0$  we take finitely many points  $\xi_1, \dots, \xi_N$  of  $\partial \Sigma_p$  satisfying

- (1)  $\xi_j$  is adjacent to  $\xi_{j-1}$ ;
- (2)  $L(\Sigma_\xi(\Sigma_p)) > \pi - \epsilon_1$  for any element  $\xi$  of the complement of  $\{\xi_1, \dots, \xi_N\}$  in  $\partial \Sigma_p$ ;
- (3) there exist positive numbers  $\delta_2 \ll \delta_3 \ll 1$  such that
  - (a) for every  $\xi \in B(\xi_j, \delta_3)$ , there exists some  $\eta \in \Sigma_p$  with  $\tilde{Z}\xi_j\xi\eta > \pi - \epsilon_1$ ;
  - (b)  $\tilde{Z}\xi_j\xi\xi_{j+1} > \pi - \epsilon_1$  for every  $\xi \in \widehat{B(\xi_j\xi_{j+1}, \delta_2)} - B(\xi_j, \delta_3) - B(\xi_{j+1}, \delta_3)$ , where  $\widehat{\xi_j\xi_{j+1}}$  is the minimizing arc joining  $\xi_j$  and  $\xi_{j+1}$  in  $\partial \Sigma_p$ ;

In view of the convergence  $(\frac{1}{r}X^3, p) \rightarrow (K_p, o_p)$ ,  $r \rightarrow 0$ , we find a small  $r$  and a finite set  $\{x^1, \dots, x^N\} \subset \partial B(p, r) \cap \partial X^3 \simeq S^1$  such that

- (1)  $x^j$  is adjacent to  $x^{j-1}$ ;
- (2) there exists a sufficiently small positive number  $\delta_4 \ll \delta_3$  such that
  - (a)  $d_{GH}(\Sigma_x(X^3), D_+^2(1)) < \tau(\epsilon_1)$  for any  $x \in \partial B(p, r) \cap \partial X^3$  in the complement of  $\delta_4 r$ -neighborhood of  $\{x^1, \dots, x^N\}$ , where  $D_+^2(1) \subset S^2(1)$  denotes a closed hemisphere;
  - (b) for every  $y \in A(x^j; \delta_4 r, \delta_3 r) \cap \partial B(p, r)$ , there exists  $z \in X^3$  with  $\tilde{Z}x^jyz > \pi - \tau(\epsilon_1)$ ;
  - (c)  $\tilde{Z}x^jxx^{j+1} > \pi - \tau(\epsilon_1)$  for every point  $x \in \widehat{B(x^jx^{j+1}, \delta_2 r)} - B(x^j, \delta_3 r) - B(x^{j+1}, \delta_3 r)$ , where  $\widehat{x^jx^{j+1}}$  is a minimizing arc in  $\partial X^3 \cap \partial B(p, r)$  joining  $x^j$  and  $x^{j+1}$ .

Take the points  $\{y^1, \dots, y^N\} \subset \partial B(p, r/2)$  with  $y^j \in px^j$ . Let us assume  $\delta_1 \ll \delta_4$ . Taking  $x_i^j, y_i^j \in M_i^4$  with  $x_i^j \rightarrow x^j$  and  $y_i^j \rightarrow y^j$  under the convergence  $M_i^4 \rightarrow X^3$ , consider the closed domains of  $M_i^4$

$$B_i^j = \{\tilde{d}_{x_i^j y_i^j} \leq \delta_2 r\} \cap \{r/2 \leq \tilde{d}_{p_i} \leq r\}.$$

Let  $\Delta_j$  denote the domain on  $\partial X^3$  bounded by  $\widehat{x^j x^{j+1}}, \widehat{x^{j+1} y^{j+1}}, \widehat{y^j y^{j+1}}$  and  $y^j x^j$ . Let  $C'_i$  be a compact domain of  $M_i^4$  which converges to  $B(\Delta_j, \delta_2 r)$ ,  $C_i$  the closure of  $C'_i - B_i^j - B_i^{j+1}$ , and  $N_i$  the closure of  $\partial C_i - B_i^j - B_i^{j+1}$ . Applying Fibration Theorem 1.2 to a neighborhood of  $\{d(\Delta_j, \cdot) = \delta_2 r\}$ , we can take such  $C'_i$  that for every  $x \in (B_i^j \cup B_i^{j+1}) \cap N_i$

$$|\angle(\xi_1(x), \xi_2(x)) - \pi/2| < \tau(r) + \tau(\delta_3) + \tau(\delta_3|\delta_2) + \tau(\delta_2, \delta_3|\epsilon_i),$$

where  $\xi_1$  and  $\xi_2$  denote the unit normal vector fields to  $\partial(B_i^j \cup B_i^{j+1})$  and  $N_i$  respectively. Thus both  $\partial B_i^j$  and  $\partial B_i^{j+1}$  meet  $N_i$  transversally, and therefore  $B_i^j \cap N_i \cap \partial B(p_i, r) \simeq S^1$ ,  $B_i^{j+1} \cap N_i \cap \partial B(p_i, r) \simeq S^1$ . It follows that

$$N_i \simeq S^1 \times I \times I, \quad N_i \cap B_i^j \simeq S^1 \times I, \quad N_i \cap B_i^{j+1} \simeq S^1 \times I.$$

We next show  $C_i \simeq D^4$ . Consider the functions  $f_i$ ,

$$g_i = \tilde{d}(x_i^j y_i^j, \cdot) - \tilde{d}(x_i^{j+1} y_i^{j+1}, \cdot),$$

and  $\tilde{d}_{p_i}$ . Note that the gradient of  $f_i$  is almost perpendicular to  $N_i$  and that  $g_i$  is regular on  $C_i$ . Set  $F_i = f_i^{-1}([0, \delta_2 r/2]) \cap g_i^{-1}(0)$  and denote by  $H_i$  the set consisting of all flow curves of the gradient of  $g_i$  contained in  $C_i$  through  $F_i$ . Clearly,

$$H_i \simeq F_i \times I.$$

Since the gradient of  $f_i$  is almost perpendicular to that of  $g_i$  on  $\{f_i \geq \delta_2 r/100\} \cap C_i$ , it follows that  $\partial(F_i \cap \partial B(p_i, r)) \simeq S^1$ . Note also that  $F_i$  is topologically a product  $F_i \simeq (F_i \cap \partial B(p_i, r)) \times I$ . It follows from the generalized Margulis lemma ([20]) that  $\pi_1(F_i) \simeq \pi_1(H_i)$  is almost nilpotent, and therefore by the orientability,  $F_i \simeq D^2 \times I$ .

It is easy to construct a smooth vector field  $V_i$  on a neighborhood of  $C_i - H_i$  such that

- (1)  $V_i = \text{grad } f_i$  outside a small neighborhood of  $\partial B_i^j \cup \partial B_i^{j+1}$ ,
- (2)  $V_i$  is tangent to  $\partial B_i^j \cup \partial B_i^{j+1}$ ,
- (3)  $f_i$  is strictly decreasing along the flow curves of  $V_i$ .

Thus we have

$$(8.2) \quad C_i \simeq H_i \simeq D^2 \times I \times I.$$

Next we show that  $B_i^j \simeq D^3 \times I$ . For the points  $x_i^j$  and  $x_i^{j-1}$  we construct a compact domain  $\hat{C}_i$  in the same way as the construction of  $C_i$ . From Fibration Theorem 1.2,  $(\partial B_i^j - C_i - \hat{C}_i) \cap \partial B(p_i, r) \simeq$

$S^1 \times I$ , which together with (8.2) determines the topological type of the boundary of the 3-manifold  $B_i^j \cap \partial B(p_i, r)$  as:

$$(8.3) \quad \partial(B_i^j \cap \partial B(p_i, r)) \simeq D^2 \cup (S^1 \times I) \cup D^2 \simeq S^2.$$

It follows from Lemma 4.9 that

$$(8.4) \quad B_i^j \cap \partial B(p_i, r) \simeq U_{\delta_3 r}(p_i, x_i^j) \simeq D^3.$$

From the standard critical point theory for  $d_{x_i^j}$ ,

$$M_i^\partial(p_i, r, \delta_2 r) \cap U_{\delta_3 r}(p_i, x_i^j) \simeq D^3.$$

Now we have a locally flat embedding  $\varphi_i : S^2 = D^2 \cup (S^1 \times I) \cup D^2 \rightarrow M_i^4$  such that  $\varphi_i(S^2) = \partial(M_i^\partial(p_i, r, \delta_2 r) \cap U_{\delta_3 r}(p_i, x_i^j))$  which lies in the interior of some 3-dimensional submanifold homeomorphic to  $D^3$  and containing  $M_i^\partial(p_i, r, \delta_2 r) \cap U_{\delta_3 r}(p_i, x_i^j)$ . By the Schoenflies theorem, it extends to an embedding  $\Phi_i : D^2 \times I \rightarrow M_i^4$  such that  $F_i(D^2 \times I) = M_i^\partial(p_i, r, \delta_2 r) \cap U_{\delta_3 r}(p_i, x_i^j)$ . This provides a compatible  $D^2$ -bundle structure on  $M_i^\partial(p_i, r, \delta_2 r) \cap U_{\delta_3 r}(p_i, x_i^j)$  over  $\partial X \cap U_{\delta_3 r}(p, x^j)$ . Thus  $M_i^\partial(p_i, r, \delta_2 r) \cap \partial B(p_i, r)$  is a compatible  $D^2$ -bundle over  $\partial X^3 \cap \partial B(p, r) \simeq S^1$ , and we conclude that

$$M_i^\partial(p_i, r, \delta_2 r) \cap \partial B(p_i, r) \simeq S^1 \times D^2.$$

□

*Proof of Lemma 8.6.* From Sublemmas 8.7 and 8.8,  $\partial B(p_i, r)$  is homeomorphic to a gluing  $S^1 \times D^2 \cup S_i(D^2)$  along their boundaries, where  $S_i(D^2)$  denotes a Seifert bundle over  $D^2$  having at most one singular orbit and  $\{x\} \times \partial D^2 \subset S^1 \times D^2$  is glued with the regular fibre of  $S_i(D^2)$  over  $x \in \partial D^2$ . Thus  $\partial B(p_i, r)$  is homeomorphic to either  $S^3$  or a lens space  $L(\mu_i, \nu_i)$ , where  $\mu_i \neq 0$ , or equivalently  $L(\mu_i, \nu_i)$  is not homeomorphic to  $S^1 \times S^2$ . Therefore in view of (8.1),  $\partial B(p_i, r)$  must be homeomorphic to  $S^3$  and  $B(p_i, r) \simeq D^4$ . In particular  $S_i(D^2)$  has no singular orbits. This completes the proof of Lemma 8.6. □

*Proof of Theorem 8.5.* We proceed in a way similar to that of Theorem 7.1 as follows. Let  $\nu$  be a small positive number and  $\delta_3 > 0$  be as in Fibration-Capping Theorem 1.2. Let  $Y$  be a closed domain of  $R_{\delta_3}^D(X^3)$  which approximates  $R_{\delta_3}^D(X^3)$  in the sense that  $B(Y, \mu) \supset R_{\delta_3}^D(X^3)$  with  $\mu \ll \nu$ . Applying Theorem 1.2 to  $Y$ , we have a closed domain  $M_{\nu i}^{\text{cap}} \subset M_i^4$  and a map  $f_{i, \text{cap}} : M_{\nu i}^{\text{cap}} \rightarrow \partial_0 Y_\nu$  such that

- (1)  $f_{i, \text{cap}}$  is a  $\tau_i$ -approximation with  $\lim \tau_i = 0$ ;
- (2)  $f_{i, \text{cap}}$  is a locally trivial  $D^2$ -bundle compatible with the  $S^1$ -bundle structure, say  $\mathfrak{h}_i$ , on  $\partial M_{\nu i}^4$ , in the sense that the boundary of  $f_{i, \text{cap}}^{-1}(x)$  coincides with the orbit coming from  $\mathfrak{h}_i$  for each  $x \in \partial_0 Y_\nu$ .

Next we construct a local  $D^2$ -bundle structure at each point of  $\partial X_\nu$  extending  $f_{i,\text{cap}}$ . The following lemma follows from the convergence  $(\frac{1}{r}X^3, p) \rightarrow (K_p, o_p)$  as  $r \rightarrow 0$  and the finiteness of  $S_\delta(D(\Sigma_p))$  for any  $\delta > 0$ . We put  $S_{\delta_3}^D(X^3) := X^3 - R_{\delta_3}^D(X^3)$ .

**Lemma 8.9.** *For any  $p \in \partial X^3 \cap S_{\delta_3}^D(X^3)$ , there exist a positive integer  $k = k(p)$ , positive numbers  $\epsilon = \epsilon_p > 0$  and  $r = r(p, \epsilon) > 0$  such that for some  $x^1, \dots, x^k \in \partial B(p, 10r) \cap \partial X^3$  and for sufficiently small  $\nu$*

- (1)  $B(p, 10r) \cap (\partial X_\nu^3 - R_{\delta_3}^D(X^3)) \subset \bigcup_{j=1}^k B(px^j, \epsilon r)$ ;
- (2)  $(B(px^j, \epsilon r) - B(px^j, \epsilon^{10}r)) \cap A(p; r/100, 10r)$  does not meet  $S_{\delta_3}^D(X^3)$  for each  $1 \leq j \leq k$ ;
- (3)  $\{B(px^j, \epsilon r) \cap A(p; r/100, 10r)\}_j$  is disjoint;
- (4)  $B(px^j, \epsilon r) \cap A(p; r/100, 10r) \cap \partial X_\nu^3$  bounds a topological disk, say  $\Delta_j(p)$ , for each  $1 \leq j \leq k$ ;
- (5)  $A(p; r/100, 10r) \cap \partial X_\nu^3$  does not meet  $S_{\delta_3}(X^3)$

Here we need

**Theorem 8.10** (Generalized Shöenflies Theorem[4]). *Let  $f : S^3 \rightarrow \mathbb{R}^4$  be a locally flat topological embedding in the sense that there is a topological embedding  $F : S^3 \times (-1, 1) \rightarrow \mathbb{R}^4$  with  $F(x, 0) = f(x)$ . Let  $E \subset \mathbb{R}^4$  denote the compact domain bounded by  $f(S^3)$ , and let  $S^3 = D^2 \times S^1 \cup S^1 \times D^2$  be the canonical identification. Then there exists a homeomorphism  $H : D^2 \times D^2 \rightarrow E$  extending  $f$ .*

Put  $I_r := \Delta_j(p) \cap \partial B(p, r)$ . We have a  $D^2$ -bundle structure over  $\partial I_r$  given by  $f_{i,\text{cap}}$ , and an  $S^1$ -bundle structure over  $I_r$  given by  $\mathfrak{h}_i$ . By (8.4) and the Generalized Shöenflies Theorem for  $\mathbb{R}^3$ , those two structures extend to a trivial  $D^2$ -bundle structure over  $I_r$ , on some three-dimensional compact submanifold, which is Gromov-Hausdorff close to a compact domain of  $\partial B(p_i, r)$ . Similarly we have a compatible  $D^2$ -bundle structure over  $I_{r/100}$ , on some three-dimensional compact submanifold, which is Gromov-Hausdorff close to a compact domain of  $\partial B(p_i, r/100)$ . On the other hand, we have a  $D^2$ -bundle structure over  $\partial(\Delta_j(p) \cap B(p, r)) - \partial B(p, r) - \partial B(p, r/100)$ , given by  $f_{i,\text{cap}}$ , and we have an  $S^1$ -bundle structure over  $\Delta_j(p) \cap B(p, r)$  given by  $\mathfrak{h}_i$ . Since these two structures are compatible on the intersection, by Generalized Shöenflies Theorem 8.10, these extend to a trivial  $D^2$ -bundle structure over  $\Delta_j(p) \cap B(p, r)$  for each  $1 \leq j \leq k$ . This also provides a  $D^2$ -bundle structure over  $\partial B(p, r/100) \cap \partial X_\nu^3$ , which extends to a trivial  $D^2$ -bundle structure over  $B(p, r/100) \cap \partial X_\nu^3$  again by Generalized Shöenflies Theorem 8.10. Thus we have constructed a  $D^2$ -bundle structure over  $\partial X_\nu^3 \cap B(p, r)$  which is compatible with  $f_{i,\text{cap}}$ .

In a way similar to the global patching argument constructing a globally defined local  $S^1$ -action in Section 7, one can patch those local  $D^2$ -bundle structures compatible with  $f_{i,\text{cap}}$  and  $\mathfrak{h}_i$  to obtain a globally defined  $D^2$ -bundle structure over  $\partial X_\nu^3$  compatible with  $f_{i,\text{cap}}$  and  $\mathfrak{h}_i$ .

Note that here we use Theorem 8.10 in place of Lemma 6.9. Since the patching argument is almost parallel, the detail is omitted. This completes the proof of Theorem 8.1.  $\square$

*Proof of Corollary 0.5.* Let  $M_i^4$  be as in Corollary 0.5. By Theorem 0.2, we have a locally smooth, local  $S^1$ -action on  $M_i^4$ . Since  $M_i^4$  is simply connected, this local  $S^1$ -action is actually a global  $S^1$ -action on  $M_i^4$ . According to Fintushel [16] together with Freedman (see [18]),  $M_i^4$  is homeomorphic to a connected sum

$$(8.5) \quad S^4 \# k_i \mathbb{C}P^2 \# \ell_i(-\mathbb{C}P^2) \# m_i(S^2 \times S^2).$$

Note that  $X^3$  is simply connected, and hence each connected component of  $\partial X^3$  is a sphere. It follows from the formula  $\chi(F(\psi_i)) = \chi(M_i^4)$  (see Theorem 10.9 in [3]) that

$$k_i + \ell_i + 2m_i + 2 \leq \#\text{Ext}(\text{int } X^3) + 2\alpha(\partial X^3).$$

$\square$

*Remark 8.11.* Let  $M_i^4$  be as in Corollary 0.5. By (8.5), there exists a locally smooth  $T^2$ -action on  $M_i^4$ . By [33], this action can be reduced to a smooth action, and again [33] implies that  $M_i^4$  is diffeomorphic to the above connected sum. Furthermore we have a sequence  $g_{i_j}$ ,  $j = 1, 2, \dots$  of metrics on  $M_i^4$  such that  $(M_i^4, g_{i_j})$  collapses to the quotient space  $(M_i^4, g_i)/T^2$  under a uniform lower sectional curvature bound, where  $g_i$  is a  $T^2$ -invariant metric on  $M_i^4$ .

## 9. CLASSIFICATION OF COLLAPSING TO NONCOMPACT THREE-SPACES WITH NONNEGATIVE CURVATURE

Let a sequence of pointed complete 4-dimensional orientable Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  collapses to a pointed complete noncompact 3-dimensional Alexandrov space  $(Y^3, y_0)$  with non-negative curvature. In this section, using the results of Sections 7 and 8, we classify the topology of a large metric ball  $B(p_i, R)$  in terms of geometric properties of  $Y^3$ . The classification result will be used in the subsequent sections of Part 1 to describe the phenomena of orientable 4-manifolds collapsing to spaces of dimension  $\leq 2$ .

Let

$$Y \supset C(0) \supset C(1) \supset \dots \supset C(k),$$

be a sequence of nonempty compact totally convex subsets of  $Y$  as in Section 2. Applying Theorem 0.2 to the convergence  $(M_i^4, p_i) \rightarrow (Y, y_0)$ , we have a locally smooth, local  $S^1$ -action  $\psi_i$  on  $B(p_i, R)$  whose orbit space is homeomorphic to  $B(y_0, R)$ , where  $R$  is a large positive number. Actually, we have such a local  $S^1$ -action on a small perturbation of  $B(p_i, R)$ , which is homeomorphic to  $B(p_i, R)$ . Let  $F_i^* := F^*(\psi_i)$ ,  $E_i^* := E^*(\psi_i)$ ,  $S_i^* := S^*(\psi_i)$  and  $\mathcal{C}_i := S_i^* - \partial Y$ . We denote by  $\pi_i : B(p_i, R) \rightarrow B(y_0, R)$  the orbit map.

Let  $\ell_i$  denote the number of components of  $E_i^*$  homeomorphic to a circle, and  $m_i$  the number of components of  $E_i^*$  homeomorphic to an interval whose closure does not meet  $F_i^*$ , and  $n_i$  the number of elements of  $F_i^* \cap \text{int } Y$ .

**Theorem 9.1.** *We have the following classification of the topology of  $B(p_i, R)$  in terms of  $\ell_i$ ,  $m_i$ ,  $n_i$  and the geometric properties of  $Y$ :*

Case A.  $Y$  has no boundary.

- I. Suppose  $\dim S = 2$ , yielding  $\dim Y(\infty) = 0$ . Then  $B(p_i, R)$  is homeomorphic to an  $I$ -bundle (resp. a trivial  $I$ -bundle) over  $S_i(S)$ , a Seifert bundle over  $S$  (resp. if  $S \simeq S^2$ ).
- II. Suppose  $\dim S = 1$ .
  - (1) If  $\dim Y(\infty) \geq 1$ , then  $B(p_i, R)$  is homeomorphic to a  $D^2$ -bundle over  $T^2$  or  $K^2$ ;
  - (2) If  $\dim Y(\infty) = 0$ , then  $B(p_i, R)$  is homeomorphic to either one of the spaces in II-(1) or an  $I$ -bundle over  $S^1 \tilde{\times} K^2$ , a  $K^2$ -bundle over  $S^1$ ;
- III. Suppose  $\dim S = 0$ .
  - (1) If  $\dim C(0) = 0$ , then  $n_i \leq 1$  :  $B(p_i, R)$  is homeomorphic to  $D^4$  if  $n_i = 1$  or to  $S^1 \times D^3$  if  $n_i = 0$ ;
  - (2) If  $\dim C(0) = 1$ , then  $m_i + n_i \leq 2$  :  $B(p_i, R)$  is homeomorphic to either one of the spaces in III-(1) if  $m_i + n_i \leq 1$ , or

$$\begin{array}{ll} (K^2 \tilde{\times} I) \times I & \text{if } m_i = 2, \\ S^2 \tilde{\times}_\omega D^2 & \text{if } (m_i, n_i) = (0, 2), \end{array}$$

where  $\omega$  is explicitly estimated in terms of the singularities of the spaces of directions at the endpoints  $\partial C(0)$ ;

- (3) Suppose  $\dim C(0) = 2$ . Then  $\ell_i \leq 1$ ,  $m_i \leq 2$ ,  $\max\{4\ell_i, 2m_i\} + n_i \leq 4$  and  $B(p_i, R)$  is homeomorphic to one of the spaces in A-III-(1) if  $(m_i, n_i) = (1, 0)$ , or  $m_i = 0$ ,  $n_i \leq 1$ , or one

in the following list:

$$\begin{aligned}
& T^2 \times D^2 \bigcup_{T^2 \times I} S^1 \times D^3, & \text{if } \ell_i = 1, m_i \leq 1, \\
& T^2 \times D^2 \bigcup_{T^2 \times I} (K^2 \tilde{\times} I) \times I, & \text{if } (\ell_i, m_i) = (1, 2), \\
& (K^2 \tilde{\times} I) \times I, & \text{if } (m_i, n_i) = (2, 0), \\
& S^1 \times D^3 \bigcup_{S^1 \times D^2} D^4, & \text{if } (m_i, n_i) = (1, 1), \\
& S^2 \tilde{\times}_{\omega_2} D^2, & \text{if } (m_i, n_i) = (0, 2), \\
& S^2 \tilde{\times}_{\omega_2} D^2 \bigcup_{S^1 \times D^2} D^4, & \text{if } (m_i, n_i) = (0, 3), \\
& S^2 \tilde{\times}_{\omega_3} D^2 \bigcup_{S^1 \times D^2} S^2 \tilde{\times}_{\omega_4} D^2, & \text{if } (m_i, n_i) = (0, 4).
\end{aligned}$$

where  $|\omega_1| \in \{0, 2\}$ ,  $0 \leq |\omega_3|, |\omega_4| \leq 4$  and  $|\omega_2|$  can be explicitly estimated in terms of the ratio of  $\pi$  and the angles of  $C(0)$  at the extremal points on  $\partial C(0)$ .

Case B.  $Y$  has nonempty connected boundary.

- I. Suppose  $\dim S = 2$ , yielding  $\dim Y(\infty) = 0$ . Then  $B(p_i, R)$  is homeomorphic to a  $D^2$ -bundle over  $S$ ;
- II. Suppose  $\dim S = 1$ .
  - (1) If  $\dim Y(\infty) \geq 1$ , then  $B(p_i, R)$  is homeomorphic to  $S^1 \times D^3$ ;
  - (2) If  $\dim Y(\infty) = 0$ , then  $B(p_i, R)$  is homeomorphic to the space in B-II-(1),  $S^1 \times (P^2 \tilde{\times} I)$ ,  $S^1 \times S^2 \times I$  or an  $I$ -bundle over  $S^1 \tilde{\times} S^2$ , the nontrivial  $S^2$ -bundle over  $S^1$ .
- III. Suppose  $\dim S = 0$ . If  $Y$  has two ends, then  $\ell_i = n_i = 0$ ,  $m_i \leq 2$  and

$$B(p_i, r) \simeq \begin{cases} a \text{ lens space} \times I & \text{if } m_i \leq 1 \\ (P^3 \# P^3) \times I & \text{if } m_i = 2. \end{cases}$$

Suppose that  $Y$  has exactly one end, and consider the maximum set  $C^*$  (possibly empty) of  $d_{\partial Y}$ . Then  $\mathcal{C}_i \subset C^*$ . If  $F_i^* \cap \text{int } Y$  is empty and  $E_i^*$  is nonempty, then  $m_i = 1$  and  $B(p_i, R)$  is homeomorphic to

$$(P^2 \tilde{\times} I) \times I.$$

In the other cases, we have the following:

- (1) If  $C^*$  is empty, then  $B(p_i, R)$  is homeomorphic to  $D^4$ ;
- (2) Suppose  $C^*$  is nonempty.
  - (i) If  $\dim Y(\infty) \geq 1$ , then  $B(p_i, R)$  is homeomorphic to either  $D^4$  or  $S^2 \tilde{\times}_{\omega} D^2$ , where  $|\omega| \in \{1, 2\}$ ;



- (ii) If  $\dim Y(\infty) = 0$ , then  $B(p_i, R)$  is homeomorphic to one of the spaces in B-III-(2)-(i),  $S^2 \tilde{\times}_{\omega_1} D^2$  if  $C^*$  is a ray, or  $S^2 \tilde{\times}_{\omega_2} D^2 \bigcup_{S^1 \times D^2} D^4$  if  $\dim C^* = 2$ , where  $\omega_1$  can be estimated in terms of singularities at an interior point of  $C^*$  and  $0 \leq |\omega_2| \leq 4$ .

Case C.  $Y$  has disconnected boundary.

- I. If  $\dim Y(\infty) \geq 1$ , then  $B(p_i, R)$  is homeomorphic to a  $D^2 \times S^2$ ;
- II. If  $\dim Y(\infty) = 0$ , then  $B(p_i, R)$  is homeomorphic to either  $S^1 \times S^2 \times I$  or an  $I$ -bundle over the nontrivial  $S^2$ -bundle  $S^1 \tilde{\times} S^2$ .

Later we need to clarify the position of the singular locus  $\mathcal{C}_i = S_i^* - \partial Y$ . The following lemma will give a general information about it.

Let  $b$  denote the Busemann function on  $Y$  used for the construction of  $S$  (i.e.,  $C(0)$  is the minimum set of  $b$ ), and let  $\mu_0$  be the minimum of  $b$ .

**Lemma 9.2.** (1)  $b$  is monotone on each component of  $\mathcal{C}_i - C(0)$ ;  
 (2) For any  $x \in Y$ ,  $d_x$  has no local minimum on  $\mathcal{C}_i - \{b \leq b(x)\}$ .  
 (3) If  $\mathcal{C}_i$  is nonempty, it must meet  $C(0)$ . If  $A_i$  denotes a component of  $E_i^*$  which is not contained in  $\partial C(0)$ , then  $\bar{A}_i$  perpendicularly meet  $C(0)$ .

Here we say that a curve  $c : [a, b] \rightarrow Y$  is *perpendicular* to a closed set  $C$  if  $c(a) \in C$  and for any  $x \in C$ , the distance  $d(x, c([a, b]))$  is realized only at  $c(a)$ .

*Proof.* First we show that  $b$  has no local maximum on  $\mathcal{C}_i - C(0)$ . Suppose that  $b$  takes a local maximum  $t_0 > \mu_0$  at a point  $p \in \mathcal{C}_i - C(0)$ , and let  $\{\xi_1, \xi_2\} := \Sigma_p(\mathcal{C}_i)$ . For any  $t > t_0$  take a point  $q \in b^{-1}(t)$  with  $d(p, q) = t - t_0$ . Then  $d_q$  restricted to  $\mathcal{C}_i$  takes a local minimum at  $p$ . However for any  $x$  with  $b(x) < t_0$ , we have  $\angle(q'_p, x'_p) > \pi/2$ , which contradicts the fact that  $\mathcal{C}_i$  is an extremal subset.

The proof of (2) is similar, and hence omitted. (2) implies that  $b$  has no global minimum on  $\mathcal{C}_i - C(0)$ . (1) follows from the previous argument. (3) follows from (2).  $\square$

First we consider the case when  $Y$  has no boundary and begin with the simplest

Case A-I.  $Y$  has no boundary and  $\dim S = 2$ .

In this case,  $Y$  is isometric to the normal bundle  $N(S)$  of  $S$ , which is flat (hence if  $S \simeq S^2$ , then  $Y$  is isometric to  $S \times \mathbb{R}$ ). Let  $S_0$  be the zero section of  $N(S)$ , and  $S_i(S_0)$  be the preimage of  $S_0$  by  $\pi_i$ , which is a Seifert bundle over  $S_0$ . Now  $B(p_i, R)$  is homeomorphic to an  $I$ -bundle over  $S_i(S_0)$ .

Case A-II.  $Y$  has no boundary and  $\dim S = 1$ .

In this case  $Y$  is isometric to a quotient  $(\mathbb{R} \times N^2)/\Lambda$  which is homeomorphic to  $S^1 \times N^2$  (resp. the nontrivial bundle  $S^1 \tilde{\times} N^2$ ) if  $Y$  is orientable (resp. if  $Y$  is non-orientable), where  $\Lambda \simeq \mathbb{Z}$ ,  $N^2 \simeq \mathbb{R}^2$  and the number  $k$  of essential singular points of  $N^2$  is at most two. If  $\dim Y(\infty) = \dim N^2(\infty) \geq 1$ ,  $k$  is at most 1. If  $\dim Y(\infty) = 0$  and  $k = 2$ , then  $N^2$  would be isometric to the double of a product  $[a, b] \times [0, \infty)$ .

Let  $k_i \leq 2$  denote the number of the singular orbits of  $\psi_i$  over the  $N^2$ -factor. If  $k_i = 0$ , it follows that  $B(p_i, R)$  is an  $S^1$ -bundle over  $B(y_0, R)$ , which is homeomorphic to either  $S^1 \times D^2$  or the twisted product  $S^1 \tilde{\times} D^2$ . Thus  $B(p_i, R)$  is a  $D^2$ -bundle over  $T^2$  or  $K^2$ .

If  $k_i = 1$ , the preimage of the  $N^2$ -factor by  $\pi_i$ , denoted  $S_i(N^2)$ , is a fibred solid torus with one singular fibre. Therefore  $B(p_i, R)$  is an  $S_i(N^2)$ -bundle over  $S^1$ , and

$$B(p_i, R) \simeq S^1 \times D^2 \times I / (x, 0) \sim (f(x), 1),$$

where  $f$  is a gluing homeomorphism of  $S^1 \times D^2$ . Consider the mapping class group  $\mathcal{M}_+(S^1 \times D^2)$  of orientation-preserving homeomorphisms of  $S^1 \times D^2$ , which can be thought of as a subgroup of  $\mathcal{M}_+(S^1 \times S^1) = SL(2, \mathbb{Z})$ . Since

$$\mathcal{M}_+(S^1 \times D^2) = \left\{ \begin{pmatrix} \pm 1 & \omega \\ 0 & \pm 1 \end{pmatrix} \mid \omega \in \mathbb{Z} \right\},$$

we may assume that  $f$  preserves the  $D^2$ -factors. Thus  $B(p_i, R)$  is a  $D^2$ -bundle over  $T^2$  or  $K^2$ .

Suppose  $k_i = 2$ . In a similar way to the above case of  $k_i = 1$ ,  $B(p_i, R)$  is an  $S(D^2 : (2, 1), (2, 1))$ -bundle over  $S^1$ , where  $S(D^2 : (2, 1), (2, 1))$  is the Seifert bundle over  $D^2$  with two singular orbits with Seifert invariants  $(2, 1)$ . Note that  $S(D^2 : (2, 1), (2, 1)) \simeq K^2 \tilde{\times} I$ . Since  $\mathcal{M}_+(K^2 \tilde{\times} I) \simeq \mathcal{M}(K^2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  ([31], [30]), in the same way as before,  $B(p_i, R)$  is homeomorphic to an  $I$ -bundle over a  $K^2$ -bundle over  $S^1$ .

*Case A-III.*  $Y$  has no boundary and  $\dim S = 0$ .

Since  $B(y_0, R) \simeq D^3$  for large  $R$  ([43]),  $\psi_i$  is an  $S^1$ -action (Lemma 3.1). Suppose first  $\dim C(0) \leq 1$ . By Lemma 2.2, this is the case if  $\dim Y(\infty) \geq 1$ . If  $\dim C(0) = 0$ , or equivalently if  $C(0) = S$ , then  $\psi_i$  has at most one fixed point. Therefore  $B(p_i, R)$  is homeomorphic to either  $D^4$  or  $S^1 \times D^3$ .

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**Lemma 9.3.** *Let  $\psi$  be an  $S^1$ -action on a compact 4-manifold  $N$  with boundary such that*

- (1) *the orbit space  $N^*$  is homeomorphic to  $D^3$  and  $F^*(\psi) \subset \text{int } N^*$ ;*

- (2) *there is exactly one component  $A^*$  of  $E^*(\psi)$  whose closure  $\bar{A}^*$  does not meet  $F^*(\psi)$ , and the Seifert invariants along  $A^*$  are  $(2, 1)$ ;*
- (3) *there is a point  $x^* \in F^*(\psi) \cap \text{int } N^*$  such that the number of the components of  $E^*(\psi)$  whose closure touch  $x^*$  is at most 1.*

*Then  $N$  is nonorientable.*

*Proof.* By Van Kampen's theorem,  $\pi_1(N) = \mathbb{Z}_2$ . Let  $p : \tilde{N} \rightarrow N$  be the universal cover and  $\sigma$  the nontrivial deck transformation.  $\tilde{N}$  has an  $S^1$ -action  $\tilde{\psi}$  induced from  $\psi$ . Note that there is no component of  $E^*(\tilde{\psi})$  whose closure does not meet  $F^*(\tilde{\psi})$ . Let  $x_1, x_2 \in \tilde{N}$  be the points over  $x^*$ , and let  $\tilde{\pi} : \tilde{N} \rightarrow \tilde{N}^*$  be the orbit map. Divide  $\tilde{N}^*$  by a proper  $\sigma$ -invariant 2-disk in  $\tilde{N}^* - F^*(\tilde{\psi}) - E^*(\tilde{\psi})$  into two 3-disks  $U^*$  and  $V^*$  in such a way that  $x_1, x_2 \in U^*$  and the other elements of  $F^*(\tilde{\psi})$  are contained in  $V^*$ . Put  $U := \tilde{\pi}^{-1}(U^*)$ . By Proposition 3.7,  $(U, \tilde{\psi})$  is equivariantly homeomorphic to  $S^2 \tilde{\times}_\omega D^2$  for some  $\omega \in \mathbb{Z}$  equipped with a canonical action, denoted  $\hat{\psi}$ , provided in Section 3. Hence we may identify  $(U, \tilde{\psi})$  with  $(S^2 \tilde{\times}_\omega D^2, \hat{\psi})$ . Let

$$S^2 \tilde{\times}_\omega D^2 = B_1 \times D_1^2 \bigcup_{f_\omega} B_2 \times D_2^2,$$

be the gluing as in Section 3. Since the  $\mathbb{Z}_2$ -action defined by  $\sigma$  is  $S^1$ -equivariant,  $\sigma$  preserves  $\partial B_1 \times D^2$  and the zero-section  $(B_1 \cup B_2) \times 0$ . Since  $p(\partial B_1 \times 0)$  lies in  $E(\psi)$ , the action of  $\sigma$  on  $(B_1 \cup B_2) \times 0$  is orientation-reversing. On the other hand, since  $\sigma$  is  $S^1$ -equivariant, the  $\sigma$ -action on the  $D^2$ -factor induced from that on  $\partial B_1 \times D^2$  must be orientation-preserving. Therefore  $\sigma$  is orientation-reversing and  $N$  must be nonorientable.  $\square$

Suppose  $\dim C(0) = 1$ , that is,  $C(0)$  is a geodesic segment. Since  $\text{Ext}(Y)$  is included in  $\partial C(0)$ , so is  $F_i^*$ .

**Lemma 9.4.** *Let  $\dim C(0) = 1$  and let  $A_i$  be a component of  $E_i^*$ .*

- (1) *If  $A_i$  is not contained in  $C(0)$ ,  $\bar{A}_i$  perpendicularly intersects  $C(0)$  with an endpoint of  $\partial C(0)$ ;*
- (2) *If  $\bar{A}_i$  does not meet  $F_i^*$ , one of the following holds:*
  - (a) *If  $A_i \cap C(0)$  contains more than one element, then  $C(0)$  is a subarc of  $A_i$ ;*
  - (b)  *$A_i$  intersects  $C(0)$  with exactly one endpoint of  $C(0)$  making an angle equal to  $\pi/2$ . In this case, the Seifert invariants of  $\psi_i$  along  $A_i$  are  $(2, 1)$ .*

*Proof.* Any point  $x \in \text{int } C(0)$  has a small neighborhood  $U_x$  such that  $U_x - C(0)$  has no essential singular point. Therefore  $A_i$  must intersect  $C(0)$  with an endpoint. Suppose  $A_i \cap C(0)$  is a point, say  $x$ . Let  $\xi_1, \xi_2$  and  $\xi$  denote the directions at  $x$  determined by  $A_i$  and  $C(0)$

respectively. Since  $A_i$  is an extremal subset, it follows that  $\angle(\xi_j, \xi) = \pi/2$ ,  $j = 1, 2$ .

**Sublemma 9.5.** (1)  $d(\xi_1, \xi_2) < \pi$ ;  
(2)  $\Sigma_x$  is isometric to the double of a geodesic triangle on  $S^2(1)$  with sidelengths  $\pi/2$ ,  $\pi/2$  and  $d(\xi_1, \xi_2)$ .

*Proof.* Suppose (1) first. Note that  $L(\Sigma_{\xi_i}(\Sigma_x)) \leq \pi$ ,  $i = 1, 2$  since  $A_i \subset E_i^*$ . Let  $\blacktriangle$  denote a closed domain bounded by  $\triangle \xi \xi_1 \xi_2$  such that the inner angle, say  $\alpha$ , of  $\blacktriangle$  at  $\xi$  is at most  $\pi$ . Let  $\alpha_i$  be the inner angle of  $\blacktriangle$  at  $\xi_i$ . From the Alexandrov convexity applied to  $\blacktriangle$ ,

$$(9.1) \quad \alpha_i \geq \pi/2.$$

Let  $\blacktriangle' := \Sigma_x - \text{int } \blacktriangle$ , and  $\alpha'$  and  $\alpha'_i$  denote the inner angles of  $\blacktriangle'$  at  $\xi$  and  $\xi_i$  respectively. From (9.1),  $\alpha'_i \leq \pi - \alpha_i \leq \pi/2$ . If  $\alpha' \leq \pi$ , using the same argument as above, we conclude  $\alpha_i = \alpha'_i = \pi/2$ . The conclusion (2) follows from the rigidity (cf. [42]).

For any  $\eta \in \xi_1 \xi_2$ ,

$$d^{\blacktriangle'}(\xi, \eta) \geq d(\xi, \eta) \geq \pi/2,$$

where  $d^{\blacktriangle'}(\xi, \eta)$  denotes the inner distance in  $\blacktriangle'$ . Let  $\eta_0$  be a farthest point of  $\xi_1 \xi_2$  from  $\xi$  with respect to  $d^{\blacktriangle'}$ . Divide  $\blacktriangle'$  by a minimal segment  $\xi \eta$  in  $\blacktriangle'$  into two triangle regions  $\blacktriangle'_1$  and  $\blacktriangle'_2$  with  $\xi_i \in \blacktriangle'_i$ . We may assume that the inner angle of  $\blacktriangle'_1$  at  $\xi$  is at most  $\pi$ . The Alexandrov convexity applied to  $\blacktriangle'_1$  shows

$$0 \geq \cos d^{\blacktriangle'}(\xi, \eta_0) = \sin d(\xi_1, \eta_0) \cos \tilde{\alpha}'_1 \geq 0,$$

where  $\tilde{\alpha}'_1 \leq \pi/2$  denotes the comparison angle for  $\alpha'_1$ . It follows that  $d^{\blacktriangle'}(\xi, \eta_0) = \pi/2$  and  $\tilde{\alpha}'_1 = \alpha'_1 = \alpha_1 = \pi/2$ . Thus  $\blacktriangle'_1$  is isometric to a triangle in  $S^2(1)$  with sidelengths  $\pi/2$ ,  $\pi/2$  and  $d(\xi_1, \eta_0)$ .

Continue the above argument for  $\blacktriangle'_2$  by taking a farthest point of  $\eta_0 \xi_2$  from  $\xi$ , finally to conclude that  $\blacktriangle'$  and  $\blacktriangle$  are isometric to a triangle in  $S^2(1)$  with sidelengths  $\pi/2$ ,  $\pi/2$  and  $d(\xi_1, \xi_2)$ .

Next we show (1). Suppose  $d(\xi_1, \xi_2) = \pi$ . Then  $K_p$  is isometric to a product  $\mathbb{R} \times K(S_\ell^1)$ , where the length  $\ell$  of the circle  $S_\ell^1$  is at most  $\pi$  because of  $A_i \subset E_i^*$ . Now consider the Busemann function  $b$  such that  $C(0)$  coincides with the minimum set of  $b$ , where we assume  $C(0) = \{b = 0\}$ . Under the convergence  $(\frac{1}{r}X, p) \rightarrow (K_p, o_p)$ ,  $\frac{1}{r}b$  converges to a convex function  $b_\infty$  on  $K_p$  with Lipschitz constant 1. Let  $C_\infty$  be the limit of  $C(0)$  under the above convergence, which is a geodesic ray in  $0 \times K(S_\ell^1)$  from  $o_p$  and is contained in  $\{b_\infty = 0\}$ . Since  $\{b_\infty = 0\}$  is a totally convex set of dimension  $\leq 2$ , it follows that  $\{b_\infty = 0\} = 0 \times K(S_\ell^1)$ . Obviously  $\{b_\infty = 1\} = \{\pm 1\} \times K(S_\ell^1)$  is disconnected while  $\{b = r\} \simeq S^2$  (see Theorem 15.12) is connected. In an obvious way, we get a contradiction.  $\square$

By Sublemma 9.5, the Seifert invariants of  $\psi_i$  along  $A_i$  are  $(2, 1)$ . If  $A_i \cap C(0)$  is more than a point, then one of  $\xi_j$  must coincide with  $\xi$  and  $C(0)$  is a subarc of  $\bar{A}_i$ .  $\square$

By Lemma 9.4, we have  $m_i + n_i \leq 2$ . Lemmas 9.3 and 9.4 together with Lemma 3.5 imply that the case  $(m_i, n_i) = (1, 1)$  does not occur.

If  $m_i = 2$ , then  $B(y_0, R) \simeq D^3$  consists of continuous one-parameter family  $D_t^2$  of mutually disjoint 2-disks each of which transversally intersects  $E_i^*$  at exactly two points. Since

$$\pi_i^{-1}(D_t^2) \simeq S(D^2; (2, 1), (2, 1)) \simeq K^2 \tilde{\times} I,$$

$B(p_i, R)$  is homeomorphic to  $(K^2 \tilde{\times} I) \times I$ , a  $D^2$ -bundle over  $K^2$  whose boundary is homeomorphic to the double  $D(K^2 \tilde{\times} I)$ .

If  $m_i = 1$ , then  $B(p_i, R)$  is homeomorphic to  $S^1 \times D^3$ .

If  $m_i = 0$ , by the above argument in the case of  $\dim C(0) = 0$ , we may assume  $F_i^* = \partial C(0)$ . In view of Lemma 3.5, it follows from Proposition 3.7 that  $B(p_i, R)$  is homeomorphic to  $S^2 \tilde{\times}_\omega D^2$ . Let  $z_j$ ,  $j = 0, 1$ , be the endpoints of  $\partial C(0)$ . Then  $|\omega|$  does not exceed

$$\max \left\{ \frac{2\pi}{L(\Sigma_{\xi_0}(\Sigma_{z_0}(Y)))}, 1 \right\} + \max \left\{ \frac{2\pi}{L(\Sigma_{\xi_1}(\Sigma_{z_1}(Y)))}, 1 \right\}$$

for some essential singular point  $\xi_j$  (if it exists) of  $\Sigma_{z_j}(Y)$  with distance  $\pi/2$  to the direction determined by  $C(0)$ .

Next suppose  $\dim C(0) = 2$ . Since there are exactly two geodesic rays starting from each point of the interior of  $C(0)$  with directions normal to  $C$ , we have a locally isometric imbedding  $f : \text{int } C(0) \times \mathbb{R} \rightarrow Y$  (see Section 14). For any subset  $B \subset C(0)$ , let  $\mathcal{N}(B)$  denote the union of geodesic rays from the points of  $B$  appearing as the limits of geodesic rays in  $f(\text{int } C(0))$  normal to  $C(0)$ .

For the definition of one-normal or two normal points in  $\partial C(0)$ , see the definition right before Proposition 14.1.

We need some geometry of the space of directions at a given one-normal point of  $\partial C(0)$ , which is described in the following lemma.

**Lemma 9.6.** *Suppose  $\dim C(0) = 2$  and a point  $x \in \partial C(0)$  be given.*

- (1) *If  $x$  is an essential singular point of  $Y$ , then it is a one-normal point ;*
- (2) *If a point in  $\partial \Sigma_x(C(0))$  is an essential singular point of  $\Sigma_x$ , then  $x$  is a one-normal point ;*
- (3) *Suppose that  $x$  is a one-normal point and there are two essential singular points of  $\Sigma_x$ ; one is in  $\partial \Sigma_x(C(0))$  and the other is not. Let  $v$  be the essential singular point in  $\Sigma_x - \partial \Sigma_x(C(0))$ .*
  - (a) *If  $L(\Sigma_x(C(0))) \leq \pi/2$ , then  $v$  satisfies*

$$L(\Sigma_v(\Sigma_x)) \geq 2L(\Sigma_x(C(0)));$$

- (b) If  $L(\Sigma_x(C(0))) > \pi/2$ , then  $v$  lies on a geodesic extension of the geodesic through  $\partial\Sigma_x(C(0))$ , and  $\Sigma_x$  is isometric to the double of a geodesic triangle in  $S^2(1)$  of sidelengths  $\pi/2$ ,  $\pi/2$  and  $\max\{d(v, \eta); \eta \in \partial\Sigma_x(C(0))\}$ .
- (4) If  $x$  is an extremal point of  $Y$  and if  $v$  is an essential singular point of  $\Sigma_x$  satisfying  $d(v, \eta) = \pi/2$  for every  $\eta \in \partial\Sigma_x(C(0))$ , then  $v$  is the normal direction to  $C(0)$ .

*Proof.* Let  $\xi_1, \xi_2 \in \Sigma_x$  be the normal directions to  $C(0)$ , and  $\eta_1, \eta_2$  be the boundary points of  $\Sigma_x(C(0))$ . Note that  $\Sigma_x$  has a geodesic triangulation which consists of four geodesic triangles with vertices  $\xi_1, \xi_2, \eta_1$  and  $\eta_2$ : two of which are isometric to a triangle in  $S^2(1)$  with sidelengths  $\pi/2, \pi/2$  and  $L(\Sigma_x(C(0)))$ . Note that the other two triangles may be degenerate biangles if  $x$  is a one-normal point. In view of this geodesic triangulation, a direct calculation with the Alexandrov convexity shows that if  $x$  is a two-normal point ( $\xi_1 \neq \xi_2$ ), then the radius of  $\Sigma_x$  is greater than  $\pi/2$ , and (1) follows. Furthermore, since  $L(\Sigma_{\eta_j}(\Sigma_x)) > \pi$  in this case, both  $\eta_1$  and  $\eta_2$  are not essential singular points of  $\Sigma_x$  yielding (2).

(4) is obvious from the Alexandrov convexity.

For a one-normal point  $x$ , let  $\xi$  be a direction at  $x$  normal to  $C(0)$ . Then  $\xi, \eta_1$  and  $\eta_2$  span two geodesic triangles of constant curvature 1. Hence (3) follows from the following sublemma.  $\square$

**Sublemma 9.7.** *Let  $\Sigma$  be a compact Alexandrov surface with curvature  $\geq 1$  and with no boundary. Suppose that there are two geodesic triangle regions of constant curvature 1 in  $\Sigma$  spanned by three points  $\xi, \eta_1$  and  $\eta_2$  such that  $d(\xi, \eta_i) = \pi/2, i = 1, 2$ . Let  $v(\neq \eta_i)$  be an essential singular point in  $\Sigma - \{\eta_1, \eta_2\}$ .*

- (1) *If  $\min d(v, \eta_i) \geq \pi/2$ , then  $L(\Sigma_v(\Sigma)) \geq 2d(\eta_1, \eta_2)$ ;*
- (2) *If  $\min d(v, \eta_i) < \pi/2$ , then  $v$  lies on a geodesic extension of a geodesic joining  $\eta_1$  and  $\eta_2$ , and the geodesics  $v\xi$  is the common edge of the two geodesic triangles spanned by  $v, \xi, \eta_i$  of constant curvature 1 and of sidelengths  $\pi/2, \pi/2$  and  $\max d(v, \eta_i)$ .*

*Proof.* If  $v = \xi$ , then clearly (1) holds. We assume  $v \neq \xi$  and  $d(v, \eta_1) \leq d(v, \eta_2)$ . Since  $v$  is an essential singular point, one of  $d(\eta_1, v)$  and  $d(\xi, v)$  is not greater than  $\pi/2$  and the other is not smaller than  $\pi/2$ . We first assume  $d(\eta_1, v) \geq \pi/2$  and  $d(\xi, v) \leq \pi/2$ . Let  $\gamma(t), 0 \leq t \leq \delta$ , be a unit speed geodesic from  $\xi$  to  $v$ , and let  $\theta_t$  denote the minimal angle between  $\gamma|_{[t, \delta]}$  and the minimal geodesics from  $\gamma(t)$  to  $\eta_1$ . Since  $d(\eta_1, \cdot)$  is concave on  $\Sigma - B(\eta_1, \pi/2)$ , it follows that  $\theta_t$  is decreasing. It follows from the assumption on the existence of two geodesic triangles spanned by  $\xi, \eta_1$  and  $\eta_2$  that

$$\angle \eta_1 v \xi \geq \pi - \theta_0 \geq \angle \eta_1 \xi \eta_2 \geq d(\eta_1, \eta_2),$$

which implies (1).

If  $d(\eta_1, v) \leq \pi/2$  and  $d(\xi, v) \geq \pi/2$ , the above argument shows that  $v$  is on a geodesic through  $\eta_1$  starting from  $\eta_2$  and  $\angle \xi v \eta_1 = \pi/2$ , and the conclusion (2) follows.  $\square$

**Example 9.8.** In  $\mathbb{R}^3$  with the usual coordinates  $(x, y, z)$ , first consider the sector  $C$  on the  $(x, y)$ -plane defined by  $C = \{0 \leq r \leq a, 0 \leq \theta \leq \alpha\}$ , where  $(r, \theta)$  denotes the polar coordinates and  $a > 0, \pi/2 > \alpha > 0$ . Let  $\eta_1$  and  $\eta_2$  be the directions of the  $(x, y)$ -plane at the origin defined by  $\partial C$ .

(1) Let  $\xi = (0, 0, 1)$  be the direction at the origin, and take a direction  $v \neq \xi$ , which is sufficiently close to  $\xi$ , such that  $\pi/2 < \angle(v, \eta_1) < \angle(v, \eta_2)$ . Let  $A$  be the union of the segment  $\{tv \mid 0 \leq t \leq 1\}$  and the ray  $\{v + (0, 0, t) \mid t \geq 0\}$ . Then the double  $Y$  of the convex hull of the union of  $C \times [0, \infty)$  and  $A$  is a complete open 3-dimensional Alexandrov space with nonnegative curvature such that  $C(0) = C$ . Note that the space of directions at the origin  $(0, 0, 0) \in Y$  is the double of the geodesic quadrangle on  $S^2(1)$  spanned by  $\eta_1, \eta_2, \xi$  and  $v$ . Note that the union of  $A$  and the segment of  $\partial C$  in the direction of  $\eta_j$  provides a quasigeodesic of  $Y$  consisting of essential singular points of  $Y$ .

(2) Take any  $\beta$  with  $\alpha < \beta < \pi$ , and let  $v$  be a direction of the  $(x, y)$ -plane at the origin defined by  $\theta = \beta$ . Let  $A$  be the union of the piece of parabola,  $\{(tv, t^2) \mid 0 \leq t \leq 1\}$ , and the ray  $\{v + (0, 0, t) \mid t \geq 1\}$ . Then the double  $Y$  of the convex hull of the union of  $C \times [0, \infty)$  and  $A$  is a complete open 3-dimensional Alexandrov space with nonnegative curvature such that  $C(0) = C$ . Note that the space of directions at the origin  $(0, 0, 0) \in Y$  is the double of the geodesic triangle on  $S^2(1)$  of sidelengths  $\pi/2, \pi/2$  and  $\beta$ . Note that the union of  $A$  and the segment of  $\partial C$  in the direction of  $\theta = 0$  provides a quasigeodesic of  $Y$  consisting of essential singular points of  $Y$ .

(1) and (2) of Example 9.8 correspond to (a) and (b) of Lemma 9.6 (3) respectively.

**Lemma 9.9.** *Suppose  $\dim C(0) = 2$ , and let  $A_i$  be an oriented component of  $E_i^*$  along which the Seifert invariants of  $\psi_i$  are given by  $(\alpha_i, \beta_i)$ .*

- (1) *Unless  $A_i = \partial C(0)$ ,  $A_i$  can meet  $\partial C(0)$  only with an arc of positive length;*
- (2) *If  $A_i$  meets  $\partial C(0)$ , then  $(\alpha_i, \beta_i) = (2, 1)$ ;*
- (3) *If  $A_i$  has no intersection with  $\partial C(0)$ , then the closure  $\bar{A}_i$  meets  $C(0)$  with exactly one point, say  $x$ , in  $\text{Ext}(C(0))$ , and  $\bar{A}_i \subset \mathcal{N}(x)$ .*
  - (a) *If  $x \in \text{int } C(0)$ , then  $x \in A_i \cap \text{Ext}(\text{int } C(0))$  and  $\alpha_i \leq \frac{2\pi}{L(\Sigma_x(C(0)))}$ ;*
  - (b) *If  $x \in \partial C(0)$ , then  $x \in F_i^*$  and  $\alpha_i \leq \frac{\pi}{L(\Sigma_x(C(0)))}$ .*

Therefore  $\mathcal{C}_i$  is included in the union of  $\partial C(0)$ , quasigeodesics starting from  $\text{Ext}(C(0)) \cap \partial C(0)$  which are perpendicular to  $C(0)$ , and the geodesic segments in  $\mathcal{N}(\text{Ext}(\text{int } C(0)))$ ;

*Proof.* Suppose that  $A_i$  meets  $\partial C(0)$  with a point  $x$ . Let  $\xi_1$  and  $\xi_2$  be directions at  $x$  determined by  $A_i$ . Suppose  $A_i \cap \partial C(0) = \{x\}$ . Since  $A_i$  is an extremal subset, in view of Lemma 9.2, one can verify that both  $\xi_1$  and  $\xi_2$  are directions normal to  $C(0)$ , yielding  $x$  being a two-normal point, a contradiction to Lemma 9.6. This argument implies that a sufficiently short subarc  $A'_i$  of  $A_i$  starting from  $x$  is contained in  $\partial C(0)$  and (1) holds true. Taking a boundary regular point  $y \in A'_i$  of  $\partial C(0)$ , we see that  $\Sigma_y$  is isometric to the spherical suspension over  $S^1_\pi$ , which yields (2).

Suppose that  $A_i$  does not meet  $\partial C(0)$ . From Lemma 9.2, the closure  $\bar{A}_i$  must meet  $C(0)$ , say at  $x$ . If  $x \in \text{int } C(0)$ ,  $\Sigma_x$  is isometric to the spherical suspension over  $S^1_\ell$  with  $\ell = L(\Sigma_x(C(0)))$ . It follows that  $x$  is not contained in  $F_i^*$ . Therefore  $x$  is an element of  $A_i \cap \text{Ext}(\text{int } C(0))$  and we obtain  $A_i \subset \mathcal{N}(x)$  yielding  $\alpha_i \leq 2\pi/\ell$ . If  $x \in \partial C(0)$ , then it must be contained in  $F_i^*$  and hence is an extremal point of  $Y$ . Let  $v$  be the direction at  $x$  determined by  $A_i$ . By Lemma 9.2,  $A_i$  is perpendicular to  $C(0)$ . It follows from Lemma 9.6 (4) that  $v$  must be the normal direction to  $C$  at  $x$ . From the definition of quasigeodesics, one can see that  $\bar{A}_i$  must be contained in  $\mathcal{N}(x)$ , by developing  $A_i$  on the Euclidean plane from a point of  $\mathcal{N}(x)$ .  $\square$

**Lemma 9.10.** *Suppose  $\dim C(0) = 2$ . Then  $\ell_i \leq 1$ .*

- (1) *If  $\ell_i = 1$ , then the circle component of  $E_i^*$  coincides with  $\partial C(0)$ , and  $Y = D(C(0) \times [0, \infty))$ ;*
- (2) *Every component of  $\bar{E}_i^* \cap \partial C(0)$  contains at most one component of  $E_i^*$ ;*

*Proof.* Lemma 9.2 yields that there is no choice for the circle component of  $E_i^*$  but  $\partial C(0)$ , and hence  $\ell_i \leq 1$ . If  $\ell_i = 1$ , Lemma 9.6 implies that each point of  $\partial C(0)$  is a one-normal point, from which (1) follows. Suppose that there are adjacent components  $A_i$  and  $A'_i$  of  $\mathcal{C}_i$  with nonempty intersection  $\bar{A}_i \cap \bar{A}'_i$  in  $F_i^*$ . It turns out that both  $A_i$  and  $A'_i$  have Seifert invariants  $(2, 1)$ , a contradiction to Lemma 3.5.  $\square$

Suppose  $\ell_i = 1$ , and let  $p : C(0) \times \mathbb{R} \rightarrow Y$  be the map naturally extending  $f : \text{int } C(0) \times \mathbb{R} \rightarrow Y$ . Let  $C_1$  be a 2-disk domain in  $\text{int } C(0)$  containing  $\text{int } C(0) \cap E_i^*$ , and  $C_2$  the closure of  $C(0) - C_1$ . Put  $W_i^j = \pi_i^{-1}(p(C_j \times \mathbb{R}))$ ,  $j = 1, 2$ . Since  $\pi_i^{-1}(p(\partial C_1 \times \mathbb{R})) \simeq T^2 \times I$ , we have

$$B(p_i, R) \simeq W_i^1 \bigcup_{T^2 \times I} W_i^2.$$

Since  $p(C_2 \times \mathbb{R}) \cap B(y_0, R)$  is homeomorphic to a tubular neighborhood of  $\partial C(0)$ , it follows from (3.8) in [16] that  $W_i^2 \simeq T^2 \times D^2$ . Obviously  $W_i^1 \simeq S^1 \times D^3$  if  $m_i \leq 1$ .



If  $m_i = 2$ , then as before  $B(p_i, R)$  is homeomorphic to  $(K^2 \tilde{\times} I) \times I$ .  
Next consider the case of  $\ell_i = 0$ . We need

**Lemma 9.11.** *Let  $Z^2$  be a nonnegatively curved compact Alexandrov surface with boundary. Let  $m$  and  $n$  denote the numbers of the extremal points in  $\text{int } Z^2$  and in  $\partial Z^2$  respectively. Then  $2m + n \leq 4$ , where the equality holds if and only if  $Z^2$  is isometric to either a form  $D(I \times \{x \geq 0\}) \cap \{x \leq a\}$ , the result of cutting of the double  $D(I^2)$  of a square  $I^2$  along the diagonals, or a rectangle  $[a, b] \times [c, d]$ .*

*Proof.* By the Gauss-Bonnet formula, we have the inequality  $2m + n \leq 4$ , where the equality holds if and only if

- $X^2$  is smooth and flat except those essential singular points of  $Z^2$ ;
- the space of directions at an essential singular point  $x$  of  $Z^2$  has length  $\pi$  (resp.  $\pi/2$ ) if  $x \in \text{int } Z^2$  (resp. if  $x \in \partial Z^2$ );
- $\partial Z^2$  is a broken geodesic with those boundary essential singular points as brerak points.

The conclusion easily follows.  $\square$

By Lemmas 9.9 and 9.11,  $2m_i + n_i \leq 4$ , where the equality holds if and only if  $C(0)$  is isometric to one of the three types described in Lemma 9.11.

If  $m_i = 2$ ,  $B(p_i, R)$  is homeomorphic to  $(K^2 \tilde{\times} I) \times I$  as above.

Suppose  $m_i = 1$ . If  $n_i = 1$ , let  $A_i$  denote the component of  $E_i^*$  whose closure does not touch  $F_i^*$ . Divide  $B(y_0, R)$  with a proper 2-disk into two 3-disk domains  $B_1$  and  $B_2$  in such a way that  $A_i \subset B_1$  and the other components of  $\mathcal{C}_i$  is contained in  $B_2$ . Clearly  $\pi_i^{-1}(B_1) \simeq S^1 \times D^3$  and  $\pi_i^{-1}(B_2) \simeq D^4$ .

If  $(m_i, n_i) = (1, 2)$ , in view of Lemmas 9.9 and 9.11 together with Lemma 3.5, we come to the situation of Lemma 9.3 contradicting to the orientability. Hence the case  $(m_i, n_i) = (1, 2)$  does not occur.

Suppose  $m_i = 0$ . In view of the argument in the case of  $\dim C(0) = 1$ , we may assume that  $n_i \geq 2$ . If  $n_i = 2$ , Proposition 3.7 and Lemma 9.9 imply that  $B(p_i, R) \simeq S^2 \tilde{\times}_\omega D^2$ , where

$$(9.2) \quad |\omega| \leq \frac{\pi}{L(\Sigma_x(C(0)))} + \frac{\pi}{L(\Sigma_y(C(0)))},$$

and  $\{x, y\} = F_i^*$ .

If  $n_i = 3$ , then  $\mathcal{C}_i$  is disconnected. Cut  $B(y_0, R)$  with a 2-disk into two 3-disk domains  $B_1$  and  $B_2$  in such a way that a connected component of  $\mathcal{C}_i$  containing only one element of  $F_i^*$  is contained in  $B_1$  and the other components are contained in  $B_2$ . Then  $\pi_i^{-1}(B_1) \simeq D^4$  and  $\pi_i^{-1}(B_2) \simeq S^2 \tilde{\times}_\omega D^2$ , where  $|\omega|$  is estimated as in (9.2).

If  $n_i = 4$ , then  $\mathcal{C}_i$  is disconnected. Cut  $B(y_0, R)$  with a 2-disk into two 3-disk domains  $B_1$  and  $B_2$  in such a way that connected components of  $\mathcal{C}_i$  containing exactly two elements of  $F_i^*$  are contained in  $B_1$  and the

other components are contained in  $B_2$ . Then by Proposition 3.7, both  $\pi_i^{-1}(B_1)$  and  $\pi_i^{-1}(B_2)$  are homeomorphic to  $S^2 \tilde{\times}_\omega D^2$ , where  $0 \leq |\omega| \leq 4$ .

This completes the proof of Theorem 9.1 in the case when  $Y$  has no boundary.

Next we consider the case when  $Y$  has nonempty connected boundary. We begin with

*Case B-I*  $\partial Y$  is connected and  $\dim S = 2$ .

By Proposition 17.1,  $Y$  is isometric to the product  $S \times [0, \infty)$ . Therefore  $Y(\infty)$  is a point, and Theorem 8.1 implies that  $B(p_i, R)$  is homeomorphic to a  $D^2$ -bundle over  $S$ .

*Case B-II*  $\partial Y$  is connected and  $\dim S = 1$ .

In this case,  $Y$  is isometric to a product  $(\mathbb{R} \times N^2)/\Lambda$ , where  $N^2$  is either homeomorphic to  $R_+^2$  or isometric to  $I \times \mathbb{R}$ , and  $\Lambda \simeq \mathbb{Z}$ . Let  $H_i$  denote the preimage of an  $N^2$ -factor by  $\pi_i$ .

Suppose  $N^2 \simeq \mathbb{R}_+^2$ . Let  $k(\leq 1)$  denote the number of the essential singular points of  $\text{int } N^2$ . If  $k = 0$ , by Theorem 8.1,  $H_i \simeq D^3$ , yielding  $B(p_i, R) \simeq S^1 \times D^3$ . Suppose  $k = 1$ . Then  $N^2$  is isometric to a form  $D([0, \infty) \times [0, \infty)) \cap \{(x, y) \mid y \leq a\}$ , which implies that  $\dim Y(\infty) = 0$  and  $H_i \simeq P^2 \tilde{\times} I$  if  $H_i$  has a singular orbit. Since  $B(p_i, R)$  is homeomorphic to an  $H_i$ -bundle over  $S^1$ , it follows from the following lemma that  $B(p_i, R) \simeq S^1 \times (P^2 \tilde{\times} I)$ .

**Lemma 9.12.** *The mapping class group  $\mathcal{M}_+(P^2 \tilde{\times} I)$  of orientation preserving homeomorphisms of  $P^2 \tilde{\times} I$  is trivial.*

*Proof.* Let  $\mathcal{M}_D(\mathbb{R}P^3)$  be the mapping class group of homeomorphisms of  $\mathbb{R}P^3$  fixing a disk. Since any orientation preserving homeomorphism  $f$  of  $P^2 \tilde{\times} I$  is isotopic to the identity on the boundary sphere, we may assume  $f = 1$  on  $\partial P^2 \tilde{\times} I$ . Therefore  $\mathcal{M}_+(P^2 \tilde{\times} I) \simeq \mathcal{M}_D(\mathbb{R}P^3)$ . The result follows from [2], [28] and [19].  $\square$

Suppose  $N^2$  is isometric to  $I \times \mathbb{R}$ . Since  $H_i \simeq S^2 \times I$ ,  $B(p_i, R)$  is an  $(S^2 \times I)$ -bundle over  $S^1$ . Since  $\mathcal{M}_+(S^2 \times I) = \mathbb{Z}_2$ ,  $B(p_i, R)$  is homeomorphic to either  $S^1 \times S^2 \times I$  or an  $I$ -bundle over  $S^1 \tilde{\times} S^2$ , the nontrivial  $S^2$ -bundle over  $S^1$ .

*Case B-III*  $\partial Y$  is connected and  $\dim S = 0$ .

If  $Y$  has two ends, it is isometric to a product  $\mathbb{R} \times Y_0^2$ , where  $Y_0^2 \simeq D^2$ . Let  $K_i$  denote the preimage of an  $Y_0^2$ -factor by  $\pi_i$ . Then  $B(p_i, R) \simeq K_i \times I$ . Note that  $Y_0^2$  has either at most one essential singular point, say  $q$ , in its interior, or isometric to  $D(I \times \{x \geq 0\}) \cap \{x \leq a\}$  for some  $I$  and  $a > 0$ . In either case,  $\ell_i = n_i = 0$  and  $m_i \leq 2$ . In the former case,  $K_i$  is homeomorphic to  $L(\mu_i, \nu_i)$ , where  $\mu_i \leq 2\pi/L(\Sigma_q(Y_0^2))$ . In

the latter case,  $K_i$  is homeomorphic to either  $P^3$  ( $m_i = 1$ ) or  $P^3 \# P^3$  ( $m_i = 2$ ).

In what follows, we assume that  $Y$  has exactly one end. In the argument below about the geometry of  $Y$ , we show that  $\partial Y$  is homeomorphic to  $\mathbb{R}^2$ .

Since we have a collar neighborhood of  $\partial Y$  (Theorem 5.14), we can apply the method of [43] to  $Y_\epsilon$  and obtain that  $B(S, R)$  is homeomorphic to  $D^3$  for a large  $R > 0$ .

Note that  $(\frac{1}{R}Y, S)$  converges to  $(K(Y(\infty), o)$  as  $R \rightarrow \infty$ , and denote by  $B_R$  the closure of  $\partial B(S, R) - B(S, R) \cap \partial Y$ . We put  $C_R := \partial Y \cap B(S, R)$ .

**Assertion 9.13.** *Assume that  $Y$  has exactly one end. Then for any sufficiently large  $R$ , both  $B_R$  and  $C_R$  are homeomorphic to  $D^2$ .*

*In particular,  $Y$  is homeomorphic to  $\mathbb{R}_+^3$ .*

*Proof.* By using the gradient flows for the distance function  $d_S$  from  $S$ , we see that for sufficiently large  $R$ ,

- (1)  $Y - B(S, R) \simeq B_R \times (0, \infty)$ ;
- (2)  $\partial Y - C_R \simeq \partial B_R \times (0, \infty)$ ,

where  $\partial B_R = \partial C_R$  is the boundary as a topological 2-manifold. Since  $Y$  has exactly one end,  $B_R$  must be connected. We assert that  $\partial B_R$  is connected. Applying the method of [43], we also have a pseudo-gradient flow for  $d_S$  on  $B(S, R) - \{S\}$  (see Section 10 of [43] for the definition of pseudo-gradient flows). For a small  $\epsilon$  with  $B(S, \epsilon) \subset \text{int } Y$  and  $\partial B(S, \epsilon) \simeq S^2$ , let  $h : B(S, R) - \text{int } B(S, \epsilon) \rightarrow \partial B(S, \epsilon)$  be the projection along the flow curves. Since  $h(B_R) \subset S^2$  is connected, if  $\partial B_R$  was disconnected,  $h(C_R)$  and hence  $C_R$  would be disconnected, yielding the disconnectivity of  $\partial Y$ , a contradiction. The former conclusion of the assertion then follows from the connectivity of both  $h(B_R)$  and  $h(C_R)$ . The latter follows easily.  $\square$

To determine the topology of  $B(p_i, R)$ , consider the distance function  $d_{\partial Y}$  from  $\partial Y$ . Let  $C^*$  be the maximum set of  $d_{\partial Y}$  (possibly empty).

**Lemma 9.14.**  $\mathcal{C}_i \subset C^*$  for sufficiently large  $i$ .

*Proof.* The proof is by contradiction. In a way similar to Lemma 9.2, we have

- $d_{\partial Y}$  is monotone on each component of  $\mathcal{C}_i - C^*$ ;
- for any  $t > 0$  less than the maximum of  $d_{\partial Y}$ , and for any  $x$  with  $d_{\partial Y}(x) \geq t$ ,  $d_x$  has no local minimum on  $\mathcal{C}_i - \{d_{\partial Y} \geq t\}$ .

Suppose that  $\mathcal{C}_i$  is not contained in  $C^*$  for any sufficiently large  $i$ . Then we have a sequence  $R_i \rightarrow \infty$  and  $S^1$ -actions  $\psi_i$  on  $B(p_i, R_i)$  extending the original actions. Since  $d_{C^*}$  has no local minimum on  $\mathcal{C}_i - C^*$ , there is a subarc  $A_i(t)$  of  $\mathcal{C}_i$  with unit speed parameter,  $0 \leq t \leq \ell_i$ ,  $\ell_i \rightarrow \infty$ , such that (i)  $d_{\partial Y}(A_i(t))$  is monotone decreasing, (ii)  $A_i(\ell_i) \in$

$\partial B(y_0, R_i) - \partial Y$ , (iii)  $A_i(t)$  does not reach  $\partial Y$ . Therefore for any  $\epsilon > 0$  there is an  $i$  and  $s \in (0, \ell_i)$  such that

$$0 \geq (d_{\partial Y} \circ A_i)'(s) = -\cos \angle(A_i'(s), (\partial Y)'_{A_i(s)}) > -\epsilon.$$

It follows that

$$\angle(A_i'(s), \partial \Sigma_{A_i(s)}(\{d_{\partial Y} \geq d_s\})) < \tau(\epsilon),$$

where  $d_s := d_{\partial Y}(A_i(s))$ . Take  $x$  with  $d_{\partial Y}(x) = d_s$  sufficiently close to  $A_i(s)$  such that  $\angle(A_i'(s), x'_{A_i(s)}) < \tau(\epsilon)$ . Now it is easy to see that  $d_x$  has a local minimum on  $\mathcal{C}_i - \{d_{\partial Y} \geq d_s\}$ , a contradiction.  $\square$

*Case (i).*  $d_{\partial Y}$  has no maximum.

By Lemma 9.14,  $\mathcal{C}_i$  is empty, and therefore  $B(p_i, r) \simeq D^2 \times D^2$ .

*Case (ii).*  $d_{\partial Y}$  has a maximum.

From now on we assume that  $\mathcal{C}_i$  is nonempty. By the concavity of  $d_{\partial Y}$ , every geodesic ray of  $Y$  starting from any point of  $C^*$  is contained in  $C^*$ . It follows that  $Y(\infty)$  is isometric to  $C^*(\infty)$ . Obviously  $S$  is isometric to a soul of  $C^*$ .

If  $\dim C^* = 2$ , as in the case of  $\dim C(0) = 2$ , for each  $x \in \text{int } C^*$  there are two minimal geodesics from  $x$  to  $\partial Y$ . Thus we have a locally isometric imbedding  $f : \text{int } C^* \times [-a, a] \rightarrow Y$ , where  $a$  is the maximum of  $d_{\partial Y}$ .

Suppose that  $F_i^* \cap \text{int } Y$  is empty and  $E_i^*$  is nonempty. We show  $m_i = 1$ . If  $m_i \geq 2$ , then  $C^*$  is isometric to  $I \times \mathbb{R}$  and  $E_i^* = \partial C^*$ . Therefore it is easy to verify that  $Y$  is isometric to  $\mathbb{R} \times (D(I \times \{x \geq 0\}) \cap \{x \leq a\})$  for some  $a > 0$ , and thus  $Y$  would have two ends, a contradiction to the hypothesis. Therefore  $m_i = 1$ .

Suppose first the special case that  $E_i^*$  is a minimal geodesic extending to a line. Then  $Y$  is isometric to a product  $\mathbb{R} \times Y_0^2$ , where  $Y_0^2 \simeq \mathbb{R}_+^2$ . Since  $\text{int } Y_0^2$  contains an essential singular point,  $Y_0^2$  is isometric to  $D(\{x, y \geq 0\}) \cap \{y \leq a\}$  for some  $a > 0$ . Since the preimage of a  $Y_0^2$ -factor by  $\pi_i$  is homeomorphic to  $P^2 \tilde{\times} I$ ,  $B(p_i, R)$  is homeomorphic to  $(P^2 \tilde{\times} I) \times I$ . Note that the Seifert invariants along  $E_i^*$  are  $(2, 1)$  and that  $C^*$  is isometric to  $\mathbb{R}_+^2$ .

Next consider the general case. By contradiction together with the argument above, we may assume that  $\dim C^* = 2$  and  $E_i^* = \partial C^*$ . It follows that the Seifert invariants along  $E_i^*$  are  $(2, 1)$ . Therefore Proposition 3.6 combined with the above argument implies that  $B(p_i, R) \simeq (P^2 \tilde{\times} I) \times I$ .

Suppose  $\dim Y(\infty) \geq 1$ . Then  $C^*$  has at most one extremal point on  $C^*$ . Since  $\text{Ext}(Y) = \text{Ext}(C^*)$ , we have  $\ell_i = 0$ ,  $m_i \leq 1$  and  $n_i \leq 1$ . It follows from the same reason as Lemma 9.10(2) that every component of  $\bar{E}_i^* \cap \partial C^*$  contains at most one component of  $E_i^*$ . Proposition 3.7

then implies that  $B(p_i, R)$  is homeomorphic to either  $D^4$  or a  $S^2 \tilde{\times}_\omega D^2$  with  $|\omega| \in \{1, 2\}$ .

Next suppose  $\dim Y(\infty) = 0$ . If  $C^*$  was a line, then  $Y$  would split as  $Y = H^2 \times \mathbb{R}$  with  $H^2 \simeq D^2$ , a contradiction to the hypothesis for  $Y$  having exactly one end. Thus  $C^*$  must be a geodesic ray if  $\dim C^* = 1$ . Let us assume that  $C^*$  is a geodesic ray. Let  $\mathbb{Z}_{\mu_i}$ ,  $0 \leq \mu_i < \infty$ , be the isotropy group at an interior point  $x$  of  $C^*$ . Letting  $K_x = \mathbb{R} \times K(S_\ell^1)$ , we obtain  $\mu_i \leq 2\pi/\ell$ . In a similar way as above, Proposition 3.7 implies that  $B(p_i, R) \simeq S^2 \tilde{\times}_{\mu_i} D^2$ .

Suppose next that  $\dim C^* = 2$ . Since we may assume that  $C^*$  has two extremal points, it is isometric to either a product  $I \times [0, \infty)$  or the double  $D(I \times [0, \infty))$ . But in the argument below, we may assume the former case. Let  $v_1, v_2 \in \partial C^*$  be the extremal points of  $C^*$ . Here we have the following four cases for  $\mathcal{C}_i$ .

- (1)  $\mathcal{C}_i$  coincides with  $\{v_1, v_2\}$ ;
- (2)  $\mathcal{C}_i$  is the geodesic segment joining  $v_1$  and  $v_2$ ;
- (3)  $\mathcal{C}_i$  consists of  $\{v_1, v_2\}$  and a geodesic segment from  $v_1$  to  $\partial B(y_0, R)$ ;
- (4)  $\mathcal{C}_i$  consists of two geodesic segments from  $v_1$  and  $v_2$  to  $\partial B(y_0, R)$ .

Let  $U$  be a small regular closed neighborhood of  $\partial Y$ . Then  $U_i := \pi_i^{-1}(U \cap B(y_0, R))$  is homeomorphic to  $D^4$ . Let  $V_i$  denote the preimage of the closure of  $B(y_0, R) - U$  by  $\pi_i$ . Then  $B(p_i, R) \simeq U_i \bigcup_{D^2 \times S^1} V_i$ . Proposition 3.7 implies that  $V_i \simeq S^2 \tilde{\times}_\omega D^2$ , where  $|\omega| \in \{0, 2, 4\}$  in the cases of (1), (2) and (4),  $|\omega| \in \{1, 3\}$  in the case of (3).

Finally we consider

*Case C.*  $Y$  has disconnected boundary.

In this case by Theorem 17.3,  $\partial Y$  consists of two connected components and  $Y$  is isometric to a product  $Z \times I$ , where  $I$  is a closed interval and  $Z$  is a component of  $\partial Y$ . Note  $\dim Y(\infty) = \dim Z(\infty)$ .

Suppose  $\dim Y(\infty) \geq 1$ . Then  $Z$  is homeomorphic to  $\mathbb{R}^2$ . Let  $k$  be the number of the essential singular points of  $Z$ , which is at most 1.

We now apply Theorem 8.1. If  $k = 0$ ,  $B(p_i, R)$  is homeomorphic to the gluing

$$(D^2 \times I) \times S^1 \bigcup (D^2 \times \partial I) \times D^2 \simeq D^2 \times S^2.$$

Note that even if  $k = 1$  there are no singular loci of the  $S^1$ -action  $\psi_i$  in  $\text{int } B(y_0, R)$ . For if there were, they would correspond to  $\{x_0\} \times I$ , where  $x_0$  is the essential singular point of  $Z$ . This contradicts Theorem 8.1 (4). Then Theorem 8.1 again implies that  $B(p_i, R) \simeq D^2 \times S^2$ .

Next suppose  $\dim Y(\infty) = 0$ . In view of the argument above, we may assume that  $Z$  is isometric to a flat cylinder or a flat Möbius strip. By Theorem 8.1, in the former case,  $B(p_i, R)$  is homeomorphic to an  $S^2$ -bundle over  $S^1 \times I$ , which is homeomorphic to  $S^1 \times S^2 \times I$ . In the latter case,  $B(p_i, R)$  is homeomorphic to an  $S^2$ -bundle over the

twisted product  $S^1 \tilde{\times} I$ , which is homeomorphic to  $I$ -bundle over the nontrivial bundle  $S^1 \tilde{\times} S^2$ .

This completes the proof of Theorem 9.1.

When a 4-manifold  $W^4$  has boundary homeomorphic to  $S^3$ , we put  $\text{Cap } W := W \cup_{S^3} D^4$

**Proposition 9.15.** *Under the same assumption of B-III in Theorem 9.1, suppose that  $n_i = 2$  and  $E_i^*$  does not meet  $\partial B(y_0, R)$ . Then  $\partial B(p_i, R) \simeq S^3$  and  $\text{Cap } B(p_i, R)$  is homeomorphic to either  $\mathbb{C}P^2 \# (\pm \mathbb{C}P^2)$  or  $S^2 \times S^2$ .*

*Proof.* Note that

$$\partial B(p_i, R) \simeq D^2 \times \partial B_R \bigcup S^1 \times B_R \simeq S^3,$$

and that  $\text{Cap } B(p_i, R)$  is simply connected and admits a local smooth  $S^1$ -action  $\tilde{\psi}_i$  whose orbit space is homeomorphic to  $B(y_0, R)$  with  $F^*(\tilde{\psi}_i) = \partial B(y_0, R) \cup F_i^*$ . It follows from Corollary 0.5 together with the fact  $\chi(\text{Cap } B(p_i, R)) = \chi(F(\tilde{\psi}_i))$  that  $\text{Cap } B(p_i, R)$  has a required topological type.  $\square$

*Remark 9.16.* (1) One can conclude that in Proposition 9.15, if  $\bar{E}_i^*$  is a segment joining the two interior fixed points, then  $\text{Cap } B(p_i, R) \simeq S^2 \times S^2$ ;  
(2) Proposition 9.15 shows that  $B(p_i, R)$  cannot be homeomorphic to a disk-bundle under the situation of B-III if  $n_i = 2$ .

## 10. COLLAPSING TO TWO-SPACES WITHOUT BOUNDARY (SPHERE FIBRE CASE)

Let a sequence of pointed complete 4-dimensional orientable Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  converge to a pointed two-dimensional Alexandrov space  $(X^2, p)$ . Throughout this section, we assume that  $p$  is an interior point of  $X^2$ .

Now consider the local convergence  $B(p_i, 2r) \rightarrow B(p, 2r)$  for a sufficiently small positive number  $r$ . By Fibration Theorem 1.2,  $A(p_i; r, 2r)$  is homeomorphic to an  $F_i$ -bundle over  $A(p; r, 2r) \simeq S^1 \times I$ , where  $F_i$  is either  $S^2$  or  $T^2$ .

By Theorem 4.1, we have sequences  $\delta_i \rightarrow 0$  and  $\hat{p}_i \rightarrow p$  such that

- (1) for any limit  $(Y, y_0)$  of  $(\frac{1}{\delta_i} M_i^4, \hat{p}_i)$ , we have  $\dim Y \geq 3$ ;
- (2)  $B(p_i, r)$  is homeomorphic to  $B(\hat{p}_i, R\delta_i)$  for every  $R \geq 1$  and large  $i$  compared to  $R$ .

Let  $S$  be a soul of  $Y$ . It follows from Lemma 4.3 and Proposition 2.4 that

$$(10.1) \quad \dim Y(\infty) \geq 1,$$

$$(10.2) \quad \dim S \leq \dim Y - 2.$$

In this section, we consider the case of the general fibre  $F_i = S^2$ . Then we have

$$(10.3) \quad \partial B(p_i, r) \simeq S^1 \times S^2.$$

We apply Theorem 9.1 to the convergence  $(\frac{1}{\delta_i} M_i^4, \hat{p}_i) \rightarrow (Y, y_0)$  under the conditions (10.1), (10.2) and (10.3). The main purpose of this section is to prove

**Theorem 10.1.** *There is a positive number  $r_p$  such that for any  $r \leq r_p$  and any sufficiently large  $i$  compared with  $r$ ,*

- (1)  $B(p_i, r) \simeq D^2 \times S^2$ ;
- (2) *there exists an  $S^2$ -bundle structure on  $B(p_i, 2r)$  compatible to the  $S^2$ -bundle structure on  $A(p_i; r, 2r)$ .*

Since  $S_\delta(X^2)$  is discrete for any  $\delta > 0$  when  $X^2$  has no boundary, together with Fibration Theorem 1.2, Theorem 10.1 yields Theorem 0.6 in the case when the general fibre is a sphere.

First we begin with

**Lemma 10.2.** *The fundamental group of  $B(p_i, r)$  is finite and cyclic for any small  $r > 0$  and any sufficiently large  $i$ .*

*Proof.* Let  $\tilde{B}(p_i, r)$  be the universal covering space of  $B(p_i, r)$  with the deck transformation group  $\Gamma_i$ . Take a sequence  $r_i \rightarrow 0$  of positive numbers satisfying  $(\frac{1}{r_i} B(p_i, r), p_i) \rightarrow (K_p, o_p)$ . We may assume that  $(\frac{1}{r_i} \tilde{B}(p_i, r), \tilde{p}_i, \Gamma_i)$  converges to a triplet  $(Z, z_0, G)$ . We assert that  $G$  is discrete and hence  $\Gamma_i$  is isomorphic to  $G$  for large  $i$ . Otherwise, we would have a sequence  $\gamma_i \neq 1 \in \Gamma_i$  converging to the identity of  $G$  under the above convergence. Applying Fibration Theorem 1.2 to a contractible ball  $B$  in  $K_p - \{o_p\}$ , we have a closed domain  $U_i$  in  $B(p_i, r)$  fibers over  $B$ . Let  $x_i \in U_i$  be a point converging to the center of  $B$ . Obviously, the geodesic loop  $c_i$  at  $x_i$  represented by  $\gamma_i$  is contained in  $U_i$ . Since  $U_i \simeq B \times S^2$ ,  $c_i$  must be null homotopic, a contradiction to  $\gamma_i \neq 1$ . Since  $Z/G$  is the flat cone  $K_p$ , it follows that  $Z$  is also a flat cone and  $G$  is a finite cyclic group.  $\square$

Now we consider

*Case A.*  $\dim Y = 4$ .

Intuitively, this is the case when the sphere fibre converges to a sphere under the rescaling  $\frac{1}{\delta_i} M_i^4$ . In fact we have

**Proposition 10.3.** *If  $\dim Y = 4$ , then*

- (1)  $S \simeq S^2$ ;
- (2)  $B(p_i, r) \simeq S^2 \times D^2$ .

*Proof.* In view of Corollary 2.7 together with (10.3) and Lemma 10.2, we have the only possibility  $\dim S = 2$ . It follows that  $S$  is homeomorphic to  $S^2$ ,  $P^2$ ,  $T^2$  or  $K^2$ . If  $S \simeq P^2$ , then  $B(p_i, r)$  must be homeomorphic to a  $D^2$ -bundle over  $P^2$ . It turns out that  $\partial B(p_i, r)$  is homeomorphic to an  $S^1$ -bundle over  $P^2$ , a contradiction to (10.3). Similarly, if  $S$  was homeomorphic to  $T^2$  or  $K^2$ , we would have a contradiction. Thus  $S$  must be homeomorphic to  $S^2$  and  $B(p_i, r)$  is homeomorphic to a  $D^2$ -bundle over  $S^2$ . In view of (10.3), this bundle must be trivial.  $\square$

Next we consider

*Case B.*  $\dim Y = 3$ .

Intuitively, this is the case when the sphere fibre collapses to a closed interval under the rescaling  $\frac{1}{\delta_i} M_i^4$  (see Propositions 10.4 and 10.5).

Let

$$Y \supset C(0) \supset C(1) \supset \cdots \supset C(k),$$

be as in Section 2. Applying Theorem 0.2 to the convergence  $(\frac{1}{\delta_i} M_i^4, \hat{p}_i) \rightarrow (Y, y_0)$ , we have a locally smooth, local  $S^1$ -action  $\psi_i$  on  $B(p_i, r) \simeq B(\hat{p}_i, R\delta_i)$  whose orbit space is homeomorphic to  $B(y_0, R)$ , where  $R$  is a large positive number. Let  $F_i^* := F^*(\psi_i)$ ,  $E_i^* := E^*(\psi_i)$ ,  $S_i^* := S^*(\psi_i)$ ,  $C_i := S_i^* - \partial Y$ ,  $\ell_i$ ,  $m_i$ ,  $n_i$  and  $\pi_i : B(\hat{p}_i, R) \rightarrow B(y_0, R)$  be as in the previous section.

**Proposition 10.4.** *If  $\dim Y = 3$  and  $Y$  has no boundary, then*

- (1)  $Y \simeq \mathbb{R}^3$ ;
- (2)  $C(0)$  is isometric to a 1-dimensional closed interval;
- (3)  $\partial C(0) = F_i^*$ . In particular,  $\text{Ext}(Y) = \partial C(0)$ ;
- (4)  $C_i$  coincides with one of the following:

$$\begin{aligned} &\partial C(0), \quad C(0), \quad \gamma_1 \cup \gamma_2 \\ &C(0) \cup \gamma_1 \cup \gamma_2, \end{aligned}$$

where  $\gamma_1$  and  $\gamma_2$  denote quasigeodesics in  $B(y_0, R)$  starting from the endpoints  $\partial C(0)$ , reaching  $\partial B(y_0, R)$ , and being perpendicular to  $C(0)$ ;

- (5)  $B(p_i, r) \simeq S^2 \times D^2$ .

*Proof.* Note  $\dim S \leq 1$ . Suppose first

*Case B-I-1.*  $Y$  has no boundary and  $\dim S = 1$ .

Case A II-(1) of Theorem 9.1 yields a contradiction to (10.3).

Therefore we have the following case:

*Case B-I-2.*  $Y$  has no boundary and  $\dim S = 0$ .



Note that  $B(y_0, R) \simeq D^3$  for large  $R$  ([43]). Since  $\dim Y(\infty) \geq 1$ , we may assume  $\dim C(0) \leq 1$  (Lemma 2.2). In view of Case A-III of Theorem 9.1 combined with (10.3), we can exclude the case of  $\dim C(0) = 0$ . Thus  $C(0)$  must be a geodesic segment. Case A-III of Theorem 9.1 implies  $(m_i, n_i) = (0, 2)$  and  $B(p_i, r) \simeq S^2 \tilde{\times}_\omega D^2$ , where  $\omega = 0$  if and only if we have the cases (a), (c), (d) and (f) in Proposition 3.7. Thus we obtain the conclusions (4) and (5).  $\square$

Next we consider the case when  $Y$  has nonempty boundary.

**Proposition 10.5.** *If  $\dim Y = 3$  and  $Y$  has nonempty boundary, then*

- (1)  *$Y$  is isometric to a product  $Z \times I$ , where  $Z \simeq \mathbb{R}^2$  has at most one essential singular point;*
- (2)  *$B(p_i, r) \simeq S^2 \times D^2$ .*

*Proof.* We first assume

*Case B-II*  $\partial Y$  is disconnected.

In this case, by Theorem 17.3,  $Y$  is isometric to the product  $Z \times I$ , where  $I$  is a closed interval and  $Z$  is a two-dimensional open Alexandrov surface with nonnegative curvature homeomorphic to  $\mathbb{R}^2$  and with  $\dim Y(\infty) = \dim Z(\infty) \geq 1$ . By Case C-I of Theorem 9.1,  $B(p_i, r)$  must be homeomorphic to  $D^2 \times S^2$ .

Next we assume the case when  $\partial Y$  is connected.

*Case B-III-1*  $\partial Y$  is connected and  $\dim S = 1$ .

In this case, by Theorem 9.1,  $B(p_i, r) \simeq S^1 \times D^3$ , which contradicts Lemma 10.2.

*Case B-III-2*  $\partial Y$  is connected and  $\dim S = 0$ .

In this case, in view of Case B-III of Theorem 9.1, we would have a contradiction to Lemma 10.2. Thus we have proved Proposition 10.5.  $\square$

*Proof of Theorem 10.1.* By Propositions 10.3, 10.4 and 10.5, we already know that  $B(p_i, r) \simeq D^2 \times S^2$ . It is known (cf. [8], [27]) that the mapping class group  $\mathcal{M}_+(S^1 \times S^2)$  of orientation preserving homeomorphisms of  $S^1 \times S^2$  is isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . It is generated by the homeomorphisms  $f = (f_1, f_2)$  and  $g$ , where  $f_1$  and  $f_2$  are orientation reversing homeomorphisms of  $S^1$  and  $S^2$  respectively and  $g$  is defined as

$$g(e^{i\theta}, x) = (e^{i\theta}, R(\theta)x), \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that  $g^2$  is isotopic to the identity since  $\pi_1 SO(3) = \mathbb{Z}_2$ . From the  $S^2$ -bundle structure on  $A(p_i; r, 2r)$ , we have a homeomorphism

$\varphi_i : \partial B(p_i, r) \rightarrow S^1 \times S^2$ . From  $B(p_i, r) \simeq D^2 \times S^2$ , we also have a homeomorphism  $\varphi'_i : \partial B(p_i, r) \rightarrow S^1 \times S^2$ . Note that

$$B(p_i, 2r) \simeq [r, 2r] \times S^1 \times S^2 \bigcup_{\varphi'_i \circ \varphi_i^{-1}} \{r\} \times D^2 \times S^2,$$

where  $\varphi'_i \circ \varphi_i^{-1}$  is understood as a homeomorphism of  $\{r\} \times S^1 \times S^2$ . Since  $\varphi'_i \circ \varphi_i^{-1}$  is isotopic to either the identity,  $f$ ,  $g$  or  $g \circ f$ , it is now easy to verify that  $B(p_i, 2r)$  admits a compatible  $S^2$ -bundle structure in either case.  $\square$

## 11. COLLAPSING TO TWO-SPACES WITHOUT BOUNDARY (TORUS FIBRE CASE)

We consider the situation of the previous section that a sequence of pointed complete 4-dimensional orientable Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  converges to a pointed 2-dimensional Alexandrov space  $(X^2, p)$ . Throughout this section, we assume that  $p$  is an interior singular point of  $X^2$ .

Now consider the local convergence  $B(p_i, 2r) \rightarrow B(p, 2r)$  for a sufficiently small positive number  $r$ . By Fibration Theorem 1.2,  $A(p_i; r, 2r)$  is homeomorphic to an  $F_i$ -bundle over  $A(p; r, 2r) \simeq S^1 \times I$ , where  $F_i$  is either  $S^2$  or  $T^2$ . In this section, we consider the case  $F_i = T^2$ . First note

$$(11.1) \quad \partial B(p_i, r) \text{ is homeomorphic to a } T^2\text{-bundle over } S^1$$

The purpose of this section is to prove the following

**Theorem 11.1.** *If  $F_i = T^2$ , then we have*

- (1)  $B(p_i, r)$  is homeomorphic to  $T^2 \times D^2$ ;
- (2) for some  $m \leq 2\pi/L(\Sigma_p)$ , an  $m$ -fold cyclic cover  $B'(p_i, r)$  of  $B(p_i, r)$  satisfies the following commutative diagram :

$$\begin{array}{ccc} B'(p_i, r) & \xrightarrow{\simeq} & T^2 \times D^2 \\ \pi_i \downarrow & & \downarrow \pi \\ B(p_i, r) & \xrightarrow{\simeq} & (T^2 \times D^2)/\mathbb{Z}_m \end{array}$$

where the diagonal  $\mathbb{Z}_m$ -action on  $T^2 \times D^2$  is free on  $T^2$ -factor and rotational on  $D^2$ -factor;

- (3) the Seifert  $T^2$ -bundle structure on  $B(p_i, r)$  in (2) is compatible with the  $T^2$ -bundle structure on  $A(p_i; r, 2r)$ .

It should be noted that there is no restriction about the  $\mathbb{Z}_m$ -action on the  $T^2$ -factor because the  $T^2$ -factor is collapsing to a point.

Since  $S_\delta(X^2)$  is discrete for any  $\delta > 0$  when  $X^2$  has no boundary, together with Fibration Theorem 1.2, Theorem 11.1 yields Theorem 0.6 in the case when the general fibre is a torus.

Let the sequences  $\delta_i \rightarrow 0$ ,  $\hat{p}_i \rightarrow p$ , the pointed complete noncompact Alexandrov space  $(Y, y_0)$  with nonnegative curvature and its soul  $S$  be as in the previous section.

**Proposition 11.2.** *Under the situation above, we have the following:*

- (1) *if  $\dim Y = 4$ , then  $S$  is isometric to a flat torus;*
- (2) *if  $\dim Y = 3$ , then*
  - (a)  *$Y$  has no boundary;*
  - (b)  *$S$  is a circle.*

Intuitively,  $\dim Y = 4$  (resp.  $\dim Y = 3$ ) occurs when a (singular) torus fibre converges to a torus (resp. a circle) under the rescaling  $\frac{1}{\delta_i}M_i^4$ . We can think of  $S$  as the limit of the “singular torus fibre”, which is invisible yet.

For the proof of Proposition 11.2, first we need

- Lemma 11.3.** (1) *If  $\dim Y = 4$ , then  $S$  is isometric to a flat torus or a flat Klein bottle;*
- (2) *If  $\dim Y = 3$ , then  $Y$  has no boundary and  $S$  is either a circle or a point with  $C(0) = I$ ;*
- (3)  *$B(p_i, r)$  is homeomorphic to a  $D^2$ -bundle over  $T^2$  or  $K^2$ .*

*Proof.* First suppose  $\dim Y = 4$ . If  $\dim S \leq 1$ , Corollary 2.7 implies

$$B(p_i, r) \simeq \begin{cases} D^4 & \text{if } \dim S = 0 \\ S^1 \times D^3 & \text{if } \dim S = 1, \end{cases}$$

which contradicts (11.1). Thus  $\dim S = 2$ , and hence  $S$  is either homeomorphic to one of  $S^2$  and  $P^2$  or isometric to a flat torus or a flat Klein bottle. By Generalized Soul Theorem 2.6,  $B(p_i, r)$  is a  $D^2$ -bundle over  $S$ . Hence if  $S$  was homeomorphic to either  $S^2$  or  $P^2$ , we would have a contradiction to (11.1). Therefore  $S$  is isometric to a flat torus or a flat Klein bottle and we obtain the conclusion.

Next suppose that  $\dim Y = 3$ . In view of (10.1) and (10.2), Theorem 9.1 shows that  $Y$  has no boundary and (2),(3) hold.  $\square$

We take a sequence  $\delta_i \ll \mu_i \rightarrow 0$  such that  $(\frac{1}{\mu_i}B(p_i, r), p_i)$  converges to  $(K_p, o_p)$ . Let  $\Gamma_i$  be the deck transformation group of the universal covering  $\tilde{B}(p_i, r) \rightarrow B(p_i, r)$ ,  $\tilde{p}_i \in \tilde{B}(p_i, r)$  a point over  $p_i$ . Passing to a subsequence, we assume that  $(\frac{1}{\mu_i}\tilde{B}(p_i, r), \tilde{p}_i, \Gamma_i)$  converges to a triplet  $(Z, z_0, G)$  with respect to the pointed equivariant Gromov-Hausdorff convergence. Note that  $Z/G$  is isometric to  $K_p$  and  $G$  contains an abelian subgroup of index  $\leq 2$  (Lemma 11.3).

**Proposition 11.4.** *Under the situation above, we have*

- (1)  *$Z$  is isometric to a product  $\mathbb{R}^2 \times N^2$  preserved by the  $G$ -action, where  $N^2$  is a flat cone;*
- (2)  *$G$  is abelian and isomorphic to  $\mathbb{R}^2 \oplus \mathbb{Z}_m$  for some integer  $m \leq 2\pi/L(\Sigma_p)$ ;*

(3)  $B(p_i, r)$  is a  $D^2$ -bundle over  $T^2$ .

*Proof.* Let  $\Gamma_i^*$  be the abelian subgroup of  $\Gamma_i$  of index  $\leq 2$ , and  $G^*$  the limit of  $\Gamma_i^*$  under the convergence  $(\frac{1}{\mu_i}\tilde{B}(p_i, r), \tilde{p}_i, \Gamma_i) \rightarrow (Z, z_0, G)$ , which is an abelian subgroup of  $G$  of index  $\leq 2$ .

**Claim 11.5.** (1) *The identity component  $G_0^*$  of  $G^*$  is isomorphic to  $\mathbb{R}^2$ ;*

(2)  *$Z$  is isometric to a  $G_0^*$ -invariant product  $\mathbb{R}^2 \times N^2$  such that  $p_2(G_0^*)$  is compact, where  $p_2 : G_0^* \rightarrow \text{Isom}(N^2)$  is the projection.*

*Proof.* Consider the norm on  $\Gamma_i^*$  defined by  $\|\gamma\| := d(\gamma\tilde{p}_i, \tilde{p}_i)/\mu_i$ . From the proof of Lemma 11.3 (3) together with  $1/\delta_i \gg 1/\mu_i$ , we can find generators  $\gamma_i, \tau_i$  of  $\Gamma_i^*$  such that  $\|\gamma_i\| \rightarrow 0$  and  $\|\tau_i\| \rightarrow 0$  as  $i \rightarrow \infty$ . Denoting by  $H_i$  the subgroup of  $\Gamma_i^*$  generated by  $\gamma_i$ , take a minimal element  $\sigma_i \in \tau_i + H_i$  with  $\|\sigma_i\| = \min\{\|\gamma\| \mid \gamma \in \tau_i + H_i\}$ . Since  $\Gamma_i^*$  is properly discontinuous,  $\sup\{d(H_i\tilde{p}_i, \sigma_i^n\tilde{p}_i) \mid n \in \mathbb{Z}\} = \infty$ . Therefore for any large  $R > 0$  there exists an  $n_i$  such that

$$(11.2) \quad R < d(H_i\tilde{p}_i, \sigma_i^{n_i}\tilde{p}_i) < R + 1.$$

Take  $g_i \in H_i$  such that  $d(g_i\tilde{p}_i, \sigma_i^{n_i}\tilde{p}_i) = d(H_i\tilde{p}_i, \sigma_i^{n_i}\tilde{p}_i)$ . Since  $\Gamma_i^*$  is abelian, it follows from homogeneity that  $(\frac{1}{\mu_i}\tilde{B}(p_i, r), g_i\tilde{p}_i, \Gamma_i^*)$  also converges to  $(Z, z_0, G^*)$ . Let  $H$  and  $K$  be the limit of  $H_i$  and the group generated by  $\sigma_i$  respectively, and let  $I_H$  and  $I_K$  denote the minimal totally convex subsets containing  $H z_0$  and  $K z_0$  respectively. Since  $H$  and  $K$  are noncompact, both  $I_H$  and  $I_K$  contains lines and the splitting theorem implies that  $I_H$  and  $I_K$  isometrically splits as  $I_H = E_1 \times \mathbb{R}$ ,  $I_K = E_2 \times \mathbb{R}$ , where  $E_j$  are compact (see [11]). Therefore we have an  $H$ -invariant line  $\ell_1$  in  $I_H$  and a  $K$ -invariant line  $\ell_2$  in  $I_K$ . (11.2) implies that  $\ell_1$  is not parallel to  $\ell_2$ . It follows that  $Z$  is isometric to a product  $\mathbb{R}^2 \times N$ .

Obviously  $G_0^*$  preserves  $\mathbb{R}^2 \times N$  and is isomorphic to the vector group  $\mathbb{R}^2$ . In particular,  $\dim N = 2$ . If  $\text{Isom}(N^2)$  is compact, clearly  $p_2(G_0^*)$  is compact. If  $\text{Isom}(N^2)$  is noncompact, then  $Z = \mathbb{R}^4$ . Let  $\Omega$  be the set of minimal displacement of  $G_0^*$ :

$$\Omega = \{x \in \mathbb{R}^4 \mid \delta_\gamma(x) = \min \delta_\gamma, \text{ for all } \gamma \in G_0^*\},$$

where  $\delta_\gamma(x) = d(\gamma x, x)$ . Then  $\Omega$  is isometric to  $\mathbb{R}^2$  and we have the decomposition  $Z = \Omega \times \mathbb{R}^2$  satisfying the required properties.  $\square$

**Claim 11.6.**  $\dim Z(\infty) = \dim Z - 1 = 3$ . *In particular,  $N^2 \simeq \mathbb{R}^2$ .*

*Proof.* Choose a sequence  $R_i \rightarrow \infty$  such that the pointed Gromov-Hausdorff distance between  $(B(\tilde{p}_i, R_i), \frac{1}{\mu_i}\tilde{B}(p_i, r), \tilde{p}_i)$  and  $(B(z_0, R_i), z_0)$  is less than  $1/R_i$ . Take  $\epsilon_i \rightarrow 0$  with  $\lim \epsilon_i R_i = \infty$  and  $\lim \epsilon_i/\mu_i = \infty$ . By  $d_{p, GH}((\epsilon_i Z, z_0), (K(Z(\infty)), o_\infty)) \rightarrow 0$ ,  $(\frac{\epsilon_i}{\mu_i}\tilde{B}(p_i, r), \tilde{p}_i, \Gamma_i)$  converges to a triplet  $(K(Z(\infty)), o_\infty, G_\infty)$ , where  $o_\infty$  is the vertex of the cone

$K(Z(\infty))$ . In a way similar to Claim 11.5, we have  $\dim G_\infty = 2$  and  $\dim K(Z(\infty)) = 4$ .  $\square$

**Lemma 11.7.** *Let  $W^n = \mathbb{R}^k \times N^{n-k}$  be an  $n$ -dimensional complete Alexandrov space with nonnegative curvature and with  $\dim N(\infty) = n - k - 1$ . Let  $\Gamma$  be a group of isometries of  $W$  isomorphic to  $\mathbb{R}^k$  preserving the splitting  $W^n = \mathbb{R}^k \times N^{n-k}$  such that  $p_1(\Gamma) \simeq \mathbb{R}^k$  and  $p_2(\Gamma)$  is compact, where  $p_1 : \Gamma \rightarrow \text{Isom}(\mathbb{R}^k)$  and  $p_2 : \Gamma \rightarrow \text{Isom}(N)$  are the projections. Then*

$$\dim \left( \frac{W}{\Gamma} \right) (\infty) = \dim N^{n-k}(\infty) - \dim p_2(\Gamma).$$

*Proof.* From the assumption,  $p_2(\Gamma)$  is isomorphic to a torus, say  $T^m$ . Let  $(\mathbb{R} \times K(N(\infty)), o, \Gamma_\infty)$  be any limit of  $(\epsilon W, w_0, \Gamma)$  as  $\epsilon \rightarrow 0$ . It is not difficult to see that  $\Gamma_\infty \simeq \mathbb{R}^k \times T^m$ . Therefore

$$\left( \frac{W}{\Gamma} \right) (\infty) = \left( \frac{K(N(\infty))}{T^m} \right) (\infty).$$

From the assumption, there is a  $p_2(\Gamma)$ -invariant point (soul) of  $N$ , say  $p_0$ . Since we have an expanding map

$$\frac{N^{n-k}(\infty)}{T^m} \rightarrow \frac{\Sigma_{p_0}(N^{n-k})}{T^m},$$

it follows from  $\dim N(\infty) = n - k - 1$  that  $\dim(N^{n-k}(\infty)/T^m) = n - k - m - 1$ . Thus, the  $T^m$ -action on  $N(\infty)$  is effective, and we have the required equality.  $\square$

Since  $\dim(Z/G_0^*)(\infty) = 1$ , it follows from Claims 11.5, 11.6 and Lemma 11.7 that  $p_2(G_0^*) = \{1\}$ , where  $p_2 : G_0^* \rightarrow \text{Isom}(N^2)$  is the projection. It follows that  $G^*$  is isomorphic to the direct sum  $\mathbb{R}^2 \oplus (G^*/G_0^*)$  which preserves the splitting  $Z = \mathbb{R}^2 \times N^2$ . Since  $N^2/(G^*/G_0^*)$  is a flat cone, so is  $N^2$ , and  $G^*/G_0^*$  must be a finite cyclic group  $\mathbb{Z}_m$  for some integer  $m$ .

Next we show (3), or equivalently  $\Gamma_i = \Gamma_i^*$ . Suppose that (3) does not hold. Since the nontrivial deck transformation of the covering  $T^2 \rightarrow K^2$  reverses the orientation of  $T^2$  and since any element of  $G$  is orientation-preserving, any nontrivial element  $g \in G - G^*$  has the property that both  $p_1(g) \in \text{Isom}(\mathbb{R}^2)$  and  $p_2(g) \in \text{Isom}(N^2)$  are orientation-reversing. Therefore  $Z/G$  must have nonempty boundary, a contradiction.

Let  $x_0 \in N^2$  be a fixed point of  $G/G_0$ -action. Since  $L(\Sigma_{x_0})/\#(G/G_0) = L(\Sigma_p)$ , it follows that  $m \leq 2\pi/L(\Sigma_p)$ . This completes the proof of Proposition 11.4.  $\square$

From Lemma 11.3 and Proposition 11.4, the proof of Proposition 11.2 is now complete.

Let  $\check{p}_i \in \tilde{B}(p_i, r)$  be a point over  $\hat{p}_i \in B(p_i, r)$ . Passing to a subsequence, we may assume that  $(\frac{1}{\delta_i}\tilde{B}(p_i, r), \check{p}_i, \Gamma_i)$  converges to a triplet  $(W, w_0, \Gamma)$ . Note that  $\Gamma$  is abelian and  $W/\Gamma = Y$ .

**Proposition 11.8.** *Under the situation above,  $W$  is isometric to a product  $\mathbb{R}^2 \times L^2$  preserved by the  $\Gamma$ -action with  $L^2 \simeq \mathbb{R}^2$  satisfying the following:*

- (1) *Let  $q_1 : \Gamma \rightarrow \text{Isom}(\mathbb{R}^2)$  and  $q_2 : \Gamma \rightarrow \text{Isom}(L^2)$  be the projections. Then  $q_1(\Gamma)$  uniformly acts as translation and  $q_2(\Gamma) \simeq \mathbb{Z}_n$  for some positive integer  $n$ .*

*Consequently,  $\Gamma$  is isomorphic to one of  $\mathbb{Z}^2$ ,  $\mathbb{R} \oplus \mathbb{Z}$  or  $\mathbb{R} \oplus \mathbb{Z} \oplus \mathbb{Z}_n$ ;*

- (2) *Letting  $p_0 \in L$  be a  $q_2(\Gamma)$ -invariant point, we have*

$$L(\Sigma_p) \leq \frac{L(\Sigma_{p_0}(L^2))}{n} \leq \frac{2\pi}{n}.$$

*Proof.* Since  $\delta_i \ll \mu_i$ , the Alexandrov convexity in nonnegative curvature yields an expanding map  $(Z, z_0) \rightarrow (W, w_0)$ , where  $Z$  is as in Proposition 11.4 (compare Lemma 4.4). In particular,  $\dim W = 4$ . From the noncompactness of  $\Gamma$ , we have a splitting  $W = \mathbb{R} \times Q^3$ .

We show the existence of  $\Gamma$ -invariant splitting  $W = \mathbb{R}^2 \times L^2$ . First assume  $\dim Y = 4$ . From Proposition 11.2,  $\Gamma \simeq \Gamma_i \simeq \mathbb{Z}^2$  and we have a splitting  $W = \mathbb{R}^2 \times L^2$  preserved by the  $\Gamma$ -action.

Next assume that  $\dim Y = 3$ , yielding  $\dim \Gamma = 1$ . Since any limit group of a nontrivial subgroup of  $\Gamma_i$  under the convergence  $\Gamma_i \rightarrow \Gamma$  is noncompact, we have  $\Gamma_0 \simeq \mathbb{R}^1$ . By Proposition 11.2, the soul of  $Y$  is a circle. It follows that  $\Gamma/\Gamma_0$  is infinite.

Now we show that  $W$  isometrically splits as  $W = \mathbb{R}^2 \times L^2$ . Otherwise, the group of isometries of  $Q^3$  is compact, and  $\Gamma$  preserves the splitting  $\mathbb{R} \times Q^3$ . Obviously  $q_1(\Gamma) = \mathbb{R}$ , where  $q_1 : \Gamma \rightarrow \text{Isom}(\mathbb{R})$  is the projection. It turns out that  $\ker q_1$  is a infinite discrete group in the compact group  $\text{Isom}(Q^3)$ . Since  $\Gamma$  is closed in  $\text{Isom}(W)$ , it is a contradiction.

Next we show that there exists a  $\Gamma$ -invariant splitting  $W = \mathbb{R}^2 \times L^2$  such that  $q_1(\Gamma) \simeq \mathbb{R} \oplus \mathbb{Z}$  and  $q_2(\Gamma)$  is compact. If  $\text{Isom}(L^2)$  is compact, it is clear. If  $\text{Isom}(L^2)$  is noncompact, then  $W = \mathbb{R}^4$ . Let  $\Gamma_1$  be a subgroup of  $\Gamma$  isomorphic to  $\mathbb{R} \oplus \mathbb{Z}$ , and let  $\Omega$  be the set of minimal displacement of  $\Gamma_1$ . Then  $\Omega = \mathbb{R}^2$  and we have the decomposition  $W = \Omega \times \mathbb{R}^2$  satisfying the required properties.

We show (1) and (2). We put  $\Lambda := q_2(\Gamma)$  for simplicity. Let  $(K(W(\infty)), o, \Gamma_\infty)$  and  $(K(L^2(\infty)), o, \Lambda_\infty)$  be any limits of  $(\epsilon W, w_0, \Gamma)$  and  $(\epsilon L^2, u_0, \Lambda)$  as  $\epsilon \rightarrow 0$  respectively, where  $K(W(\infty)) = \mathbb{R}^2 \times K(L^2(\infty))$ . Note that  $q_1(\Gamma_\infty) = \mathbb{R}^2$ . It follows that

$$K(Y(\infty)) = \frac{\mathbb{R}^2 \times K(L^2(\infty))}{\Gamma_\infty} = \frac{K(L^2(\infty))}{\Lambda_\infty}.$$

Recall that we have an expanding map

$$K(\Sigma_p) \rightarrow K(Y(\infty)) = \frac{K(L^2(\infty))}{\Lambda_\infty}.$$

It follows that  $\Lambda_\infty$  is finite and therefore  $\Lambda \simeq \Lambda_\infty$ . Taking a  $\Lambda$ -invariant point  $p_0 \in L$ , we have an injective homomorphism  $\rho : \Lambda \rightarrow SO(2)$ . Note that we also have an expanding map

$$\frac{K(L^2(\infty))}{\Lambda_\infty} \rightarrow \frac{K_{p_0}(L^2)}{\Lambda},$$

which concludes that

$$L(\Sigma_p) \leq \frac{L(\Sigma_{p_0}(L^2))}{n} \leq \frac{2\pi}{n}.$$

Finally remark that if  $\Gamma$  has no torsion,  $\Gamma$  is isomorphic to either  $\mathbb{Z}^2$  or  $\mathbb{R} \oplus \mathbb{Z}$ , and if  $\Gamma$  has a torsion,  $\Gamma$  is isomorphic to  $\mathbb{R} \oplus \mathbb{Z} \oplus \mathbb{Z}_n$ .  $\square$

**Lemma 11.9.** *A finite covering space of  $B(p_i, r)$  is homeomorphic to  $T^2 \times D^2$ .*

*Proof.* Put  $\Gamma' := \ker(q_2)$  and take the subgroup  $\Gamma'_i \subset \Gamma_i$  such that

- (1)  $\Gamma'_i$  converges to  $\Gamma'$  under the convergence  $(\frac{1}{\delta_i}\tilde{B}(p_i, r), \check{p}_i, \Gamma_i) \rightarrow (W, w_0, \Gamma)$ ;
- (2)  $\Lambda_i := \Gamma_i/\Gamma'_i \simeq \Gamma/\Gamma' \simeq \Lambda \simeq \mathbb{Z}_n$ .

We consider the quotient  $B'(p_i, r) := \tilde{B}(p_i, r)/\Gamma'_i$ . Let  $\check{p}_i \in B'(p_i, r)$  be a point over  $\hat{p}_i$ . Lifting the  $d_{\hat{p}_i}$ -gradient flow on  $B(p_i, r) - B(\hat{p}_i, R\delta_i)$  to that on  $B'(p_i, r) - B(\check{p}_i, R\delta_i)$  combined with  $d_{\check{p}_i}$ -gradient flow, we obtain

$$B'(p_i, r) \simeq B(\check{p}_i, R\delta_i)$$

(see Theorem 4.1).

First assume  $\dim Y = 4$ , and note that  $(\frac{1}{\delta_i}B'(p_i, r), \check{p}_i, \Lambda_i)$  converges to  $(T^2 \times L^2, w'_o, \Lambda)$ . By Stability Theorem 1.5,  $B'(p_i, r) \simeq B(\check{p}_i, R\delta_i)$  is homeomorphic to  $T^2 \times D^2$  for a sufficiently large  $R$ .

Next assume  $\dim Y = 3$ . Then  $(\frac{1}{\delta_i}B'(p_i, r), \check{p}_i, \Lambda_i)$  converges to  $(\mathbb{R}^2/\Gamma' \times L^2, w'_o, \Lambda)$ , where  $\mathbb{R}^2/\Gamma' \simeq S^1$ . Recall that

$$L(\Sigma_{p_0}(L^2))/n \geq L(\Sigma_p(X^2)),$$

for the  $\Lambda$ -fixed point  $p_0 \in L^2$  (see Proposition 11.8). By Theorem 7.1,  $B'(p_i, r)$  is homeomorphic to either an  $S^1$ -bundle or a Seifert  $S^1$ -bundle over  $S^1 \times D^2$ . Thus  $B'(p_i, r)$  is homeomorphic to either  $T^2 \times D^2$  or a  $S_i(L^2)$ -bundle over  $S^1$ , denoted  $S^1 \tilde{\times} S_i(L^2)$ , where  $S_i(L^2)$  is a Seifert  $S^1$ -bundle over  $L^2$  with exactly one singular orbit. Suppose that  $B'(p_i, r)$  is not homeomorphic to  $T^2 \times D^2$ , and consider  $I \times S_i(L^2) \subset S^1 \tilde{\times} S_i(L^2) = B'(p_i, r)$ , where  $I \subset S^1$  is a closed interval. For a point  $q_i \in S_i(L^2)$  converging to the singular point locus  $\in L^2$  and for any fixed  $t_0$  in the interior of  $I$ , put  $\bar{p}_i := (t_0, q_i) \in I \times S_i(L^2) \subset B'(p_i, r)$ . Take  $\nu_i \rightarrow 0$  such that  $(\frac{1}{\nu_i}(I \times S_i(L^2)), \bar{p}_i)$  converges to  $(\mathbb{R} \times K(\Sigma_{p_0}(L^2)), o)$ , and let  $\tilde{S}_i(L^2)$

denotes the universal cover of  $S_i(L^2)$  with the deck transformation group  $H_i \subset \Gamma_i$ . We may assume that  $(\frac{1}{\nu_i}(I \times \tilde{S}_i(L^2)), \bar{p}_i, H_i)$  converges to a triplet  $(\mathbb{R} \times \mathbb{R} \times L_1^2, o, H)$ . From our assumption on the existence of singular orbit in  $S_i(L^2)$ , it follows that  $H \simeq \mathbb{R} \oplus \mathbb{Z}_{n_1}$  for some  $n_1 > 1$ . Note that  $n_1$  is related with the Seifert invariants of the singular orbit of  $S_i(L^2)$  (see Proposition 4.3 in [43]). Take  $H'_i \subset H_i$  such that

- (1)  $(\frac{1}{\nu_i}(I \times \tilde{S}_i(L^2)), \bar{p}_i, H'_i)$  converges to  $(\mathbb{R} \times \mathbb{R} \times L_1^2, o, H_0)$ ;
- (2)  $H_i/H'_i \simeq \mathbb{Z}_{n_1}$ .

Note that  $K(\Sigma_{p_1}(L_1^2))/\mathbb{Z}_{n_1} = K(\Sigma_{p_0}(L^2))$  for a  $\mathbb{Z}_{n_1}$ -invariant point  $p_1 \in L_1^2$ . Therefore

$$L(\Sigma_{p_1}(L_1^2)) \geq n_1 n L(\Sigma_p(X^2)).$$

Repeating this procedure finitely many times, we obtain finite sequences of complete nonnegatively curved surfaces  $L^2, L_1^2, \dots, L_k^2$ , Seifert bundles  $S_i(L^2), S_i(L_1^2), \dots, S_i(L_k^2)$  and groups  $H_i \supset H_i^{(1)} \supset \dots \supset H_i^{(k)}$  with  $H_i^{(j)} = \pi_1(S_i(L_j^2))$  such that the last  $S_i(L_k^2)$  contains no singular orbits. Letting  $\rho_i : \Gamma'_i \rightarrow \pi_1(S_i(L^2)) \oplus \mathbb{Z}$  be an isomorphism, put  $\Gamma''_i := \rho_i^{-1}(H_i^{(k)} \oplus \mathbb{Z})$ . Then  $\tilde{B}(p_i, r)/\Gamma''_i$  is homeomorphic to an  $S^1$ -bundle over  $S^1 \times D^2$ , which is homeomorphic to  $T^2 \times D^2$ .  $\square$

*Proof of Theorem 11.1.* Since  $\partial B(p_i, r)$  is orientable, it follows from Lemma 11.9 that  $\partial B(p_i, r) \simeq T^3$  (see [40]), and therefore  $B(p_i, r)$  has Euler number 0 as a  $D^2$ -bundle over  $T^2$  and is homeomorphic to  $T^2 \times D^2$ .

We put a compatible Seifert  $T^2$ -fibre structure on  $B(p_i, r)$  as follows: We again go back to the convergence  $(\frac{1}{\mu_i}\tilde{B}(p_i, r), \tilde{p}_i, \Gamma_i) \rightarrow (\mathbb{R}^2 \times N^2, z_0, G)$  in Proposition 11.4, and consider  $B'(p_i, r) = \tilde{B}(p_i, r)/\Gamma'_i$ , where  $\Gamma'_i$  is a subgroup of  $\Gamma_i$  converging to  $G_0$  under the above convergence. Let  $A'(p_i; r, 2r) := \pi_i^{-1}(A(p_i; r, 2r))$ , where  $\pi_i : B'(p_i, 2r) \rightarrow B(p_i, 2r)$  is the covering projection. Since  $(\frac{1}{\mu_i}B'(p_i, r), p'_i, \Gamma_i/\Gamma'_i)$  converges to  $(N^2, \bar{z}_0, G/G_0)$ , by Equivariant Fibration Theorem 18.4, we have a  $\mathbb{Z}_m$ -equivariant  $T^2$ -bundle  $A'(p_i; r, 2r) \rightarrow A(\bar{z}_0; r, 2r)$  and hence a  $\mathbb{Z}_m$ -equivariant homeomorphism  $A'(p_i; r, 2r) \simeq T^2 \times A(\bar{z}_0; r, 2r)$  where  $\mathbb{Z}_m = \Gamma_i/\Gamma'_i$  acts diagonally on  $T^2 \times A(\bar{z}_0; r, 2r)$ ; freely on the  $T^2$ -factor and rotationally on  $A(\bar{z}_0; r, 2r)$ . Therefore  $A(p_i; r, 2r)$  is homeomorphic to the diagonal quotient  $(T^2 \times A(\bar{z}_0; r, 2r))/\mathbb{Z}_m$ . Finally we fill  $B(p_i, r)$  with an obvious gluing by the  $T^2$ -fibred  $T^2 \times D^2 \simeq (T^2 \times D^2(r))/\mathbb{Z}_m$ , of the same type of  $\mathbb{Z}_m$ -quotient as  $(T^2 \times A(\bar{z}_0; r, 2r))/\mathbb{Z}_m$ . We therefore obtain  $B(p_i, 2r) \simeq (T^2 \times D^2)/\mathbb{Z}_m$ . This completes the proof of Theorem 11.1.  $\square$

## 12. COLLAPSING TO TWO-SPACES WITH BOUNDARY

In this section, we prove Theorem 0.7. First we define the fibre space  $\mathcal{F}(X)$  stated there.



Let  $X$  be a compact 2-dimensional topological manifold with boundary, and let  $F$  denote  $S^2$  or  $T^2$ . Let  $\mathcal{F}_{\text{int}}(X)$  denote either an  $S^2$ -bundle over  $X$  (if  $F = S^2$ ) or a Seifert  $T^2$ -bundle over  $X$  (if  $F = T^2$ ) which can be thought of as a main building block of  $\mathcal{F}(X)$ . The building blocks of the complement of  $\mathcal{F}_{\text{int}}(X)$  are constructed as follows. Consider the following families of compact orientable 3- or 4-manifolds with boundary:

- (1) The family  $\mathcal{A}_s$  consists of  $D^3$  and a twisted  $I$ -bundle  $P^2 \tilde{\times} I$  over the projective plane  $P^2$ ;
- (2) The family  $\mathcal{A}_t$  consists of  $S^1 \times D^2$  and a twisted  $I$ -bundle  $K^2 \tilde{\times} I$  over the Klein bottle  $K^2$ ;
- (3) The family  $\mathcal{B}_s$  consists of  $D^4$  and  $D^2$ -bundles over  $S^2$  with Euler numbers  $\pm 1$ ,  $\pm 2$ , and of a  $D^2$ -bundle  $P^2 \tilde{\times}_0 D^2$  over  $P^2$  with Euler number 0;
- (4) The family  $\mathcal{B}_t$  consists of  $D^4$ ,  $S^1 \times D^3$  and  $D^2$ -bundles over  $S^2$ ,  $P^2$  or  $K^2$ .

Note that  $P^2 \tilde{\times}_0 D^2$  can be characterized as the  $D^2$ -bundle over  $P^2$  with boundary homeomorphic to  $P^3 \# P^3$ .

Let successive points (possibly empty)  $q_0, \dots, q_{k-1}$  of  $C$  be associated with each component  $C$  of  $\partial X$ . Let  $r$  represent  $s$  (resp.  $t$ ) if  $F = S^2$  (resp. if  $F = T^2$ ). Suppose that an element  $Q_{\alpha-1, \alpha} \in \mathcal{A}_r$ , called a *section*, is associated with each edge  $\widehat{q_{\alpha-1} q_\alpha}$  of  $C$ , and that an element  $R_\alpha \in \mathcal{B}_r$ , called a *connecting part*, the part connecting  $Q_{\alpha-1, \alpha}$  and  $Q_{\alpha, \alpha+1}$ , is associated with each point  $q_\alpha$ , so as to satisfy the following:

- (1)  $\partial R_\alpha$  is a gluing of  $Q_{\alpha-1, \alpha}$  and  $Q_{\alpha, \alpha+1}$  along their boundaries;
- (2) Let  $I = [0, 1]$  and let  $\mathcal{F}_{\text{cap}}(C)$  be an identification space of  $R_\alpha$ ,  $Q_{\alpha-1, \alpha} \times I$ ,  $0 \leq \alpha \leq k-1 \pmod{k}$ , where  $Q_{\alpha-1, \alpha} \times 1$  and  $Q_{\alpha, \alpha+1} \times 0$  are glued with  $Q_{\alpha-1, \alpha} \subset \partial R_\alpha$  and  $Q_{\alpha, \alpha+1} \subset \partial R_\alpha$  respectively;
- (3) Note that  $\partial \mathcal{F}_{\text{cap}}(C)$  has an  $F$ -bundle structure over  $C$ . Then  $\partial \mathcal{F}_{\text{cap}}(C)$  is required to be fibre-wise homeomorphic to the component of  $\partial \mathcal{F}_{\text{int}}(X)$  corresponding to  $C$ .

Letting  $\mathcal{F}_{\text{cap}}(\partial X)$  denote the disjoint union  $\amalg_C \mathcal{F}_{\text{cap}}(C)$ , we can glue  $\mathcal{F}_{\text{int}}(X)$  and  $\mathcal{F}_{\text{cap}}(\partial X)$  along their boundary fibres, which is denoted by

$$\mathcal{F}(X) := \mathcal{F}_{\text{int}}(X) \bigcup \mathcal{F}_{\text{cap}}(\partial X).$$

The set of points  $\{q_\alpha\}$  are called the *break points* of  $\partial X$  associated with the fibre space  $\mathcal{F}(X)$ . Figure 4 illustrates the decomposition in  $\mathcal{F}(X)$ .

From the construction,  $\mathcal{F}(X)$  has a singular fibre structure over  $X$ . More explicitly, there is a continuous surjective map  $f : \mathcal{F}(X) \rightarrow X$  such that  $f$  restricted to  $\text{int } X$  is either an  $S^2$ -bundle or a Seifert  $T^2$ -bundle and  $f$  restricted to  $\partial X$  is a singular fibration whose fibres are ones of a point,  $S^1$ ,  $S^2$ ,  $P^2$  and  $K^2$  determined by the topological types

of sections and connecting parts involved in  $\mathcal{F}_{\text{cap}}(\partial X)$  (see the figures (1)  $\sim$  (10) after Lemma 12.8).

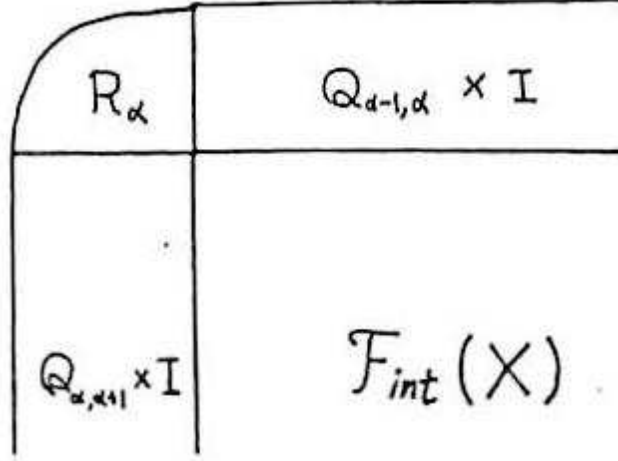


FIGURE 4.

Now Theorem 0.7 is reformulated in more detail as follows.

**Theorem 12.1.** *Suppose that a sequence of 4-dimensional closed orientable Riemannian manifolds  $M_i^4$  collapses to a two-dimensional compact Alexandrov space  $X$  with boundary under  $K \geq -1$ . Then  $M_i^4$  is homeomorphic to a fibre space  $\mathcal{F}(X)$  defined above such that the break points of  $\partial X$  are contained in  $\text{Ext}(X)$ .*

To prove Theorem 12.1, we need to understand the topology of a small metric ball in a collapsed 4-manifold near a boundary point in the limit space. For this reason, we again concentrate on a local problem, and consider the following situation that a sequence of pointed complete 4-dimensional orientable Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  converge to a pointed 2-dimensional Alexandrov space  $(X^2, p)$ . Throughout the rest of this section, we assume that  $p$  is a boundary point of  $X^2$ .

We consider the local convergence  $B(p_i, r) \rightarrow B(p, r)$  for a sufficiently small positive number  $r$ . By Theorem 4.1, we have sequences  $\delta_i \rightarrow 0$  and  $\hat{p}_i \rightarrow p$  such that

- for any limit  $(Y, y_0)$  of  $(\frac{1}{\delta_i} M_i, \hat{p}_i)$ , we have  $\dim Y \geq 3$ ;
- $B(p_i, r)$  is homeomorphic to  $B(\hat{p}_i, R\delta_i)$  for every  $R \geq 1$  and large  $i$  compared to  $R$ .

In the sequel, we shall study the topology of  $B(p_i, r)$  to prove Theorem 0.7. For a fixed small number  $r > 0$ , let  $\nu$  be a sufficiently small positive number with  $\nu/(\tan \theta_0/2) < r/2$ , where  $\theta_0 = L(\Sigma_p)(\leq \pi)$ . For arbitrary fixed  $r_0 \in (\nu/(\tan \theta_0/2), r/2)$ , applying Fibration-Capping

Theorem 1.2 to  $B(p, r) \cap X_\nu$  and  $A(p; r_0, r) \cap X_\nu$ , we have decompositions

$$(12.1) \quad B(p_i, r) = B_{\text{int}}(p_i, r) \cup B_{\text{cap}}(p_i, r),$$

$$(12.2) \quad A(p_i; r_0, r) = A_{\text{int}}(p_i; r_0, r) \cup A_{\text{cap}}(p_i; r_0, r),$$

together with compatible fibre bundle maps

$$(12.3) \quad f_{i,\text{int}} : B_{\text{int}}(p_i, r) \rightarrow B(p, r) \cap X_\nu,$$

$$(12.4) \quad f_{i,\text{cap}} : A_{\text{cap}}(p_i; r_0, r) \rightarrow A(p; r_0, r) \cap \partial X_\nu.$$

Let us denote by  $F_i$  the fibre, either  $S^2$  or  $T^2$ , of  $f_{i,\text{int}}$ . Note that  $f_{i,\text{cap}}$  has two types of fibres, one of which is denoted  $F_{i,\text{cap}}$ .

**Theorem 12.2.** *Under the situation above the following holds:*

- (1) *If  $F_i = S^2$ , then*
  - (a)  *$F_{i,\text{cap}}$  is homeomorphic to either  $D^3$  or the twisted product  $P^2 \tilde{\times} I$ ;*
  - (b)  *$B(p_i, r)$  is homeomorphic to either  $D^4$ ,  $S^2 \tilde{\times}_\omega D^2$  with  $|\omega| \in \{1, 2\}$ , or  $P^2 \tilde{\times}_0 D^2$ ;*
  - (c) *if  $B(p_i, r)$  is homeomorphic to  $P^2 \tilde{\times}_0 D^2$ , then*
    - (i) *the universal cover  $\tilde{B}(p_i, r)$  of  $B(p_i, r)$  satisfies the following commutative diagram :*

$$\begin{array}{ccc} \tilde{B}(p_i, r) & \xrightarrow{\cong} & S^2 \times D^2 \\ \pi_i \downarrow & & \downarrow \pi \\ B(p_i, r) & \xrightarrow{\cong} & (S^2 \times D^2)/\mathbb{Z}_2, \end{array}$$

*where the diagonal  $\mathbb{Z}_2$ -action is free on the  $S^2$ -factor and by reflection on the  $D^2$ -factor;*

- (ii) *the singular  $S^2$ -bundle structure on  $B(p_i, r)$  in (1)-(c)-(i) is compatible with the fibre structures on  $B_{\text{int}}(p_i, r)$  and  $A_{\text{cap}}(p_i; r_0, r)$  induced from  $f_{i,\text{int}}$  and  $f_{i,\text{cap}}$  respectively.*
- (2) *If  $F_i = T^2$ , then*
  - (a)  *$F_{i,\text{cap}}$  is homeomorphic to either  $S^1 \times D^2$  or the twisted product  $K^2 \tilde{\times} I$ ;*
  - (b)  *$B(p_i, r)$  is homeomorphic to either  $D^4$ ,  $S^1 \times D^3$ , or a  $D^2$ -bundle over  $S^2$ ,  $P^2$  or  $K^2$ ;*
  - (c) *if  $B(p_i, r)$  is homeomorphic to a  $D^2$ -bundle over  $K^2$ , then*
    - (i) *the double cover of  $B(p_i, r)$  is homeomorphic to  $T^2 \times D^2$ ;*

- (ii) for some  $m \leq \pi/L(\Sigma_p)$ , an  $2m$ -fold dihedral cover  $B'(p_i, r)$  of  $B(p_i, r)$  satisfies the following commutative diagram :

$$\begin{array}{ccc} B'(p_i, r) & \xrightarrow{\simeq} & T^2 \times D^2 \\ \pi_i \downarrow & & \downarrow \pi \\ B(p_i, r) & \xrightarrow{\simeq} & (T^2 \times D^2)/D_{2m}, \end{array}$$

where the diagonal dihedral  $D_{2m}$ -action is free on the  $T^2$ -factor and the action on the  $D^2$ -factor is generated by a rotation and a reflection ;

- (iii) the (generalized) Seifert  $T^2$ -bundle structure on  $B(p_i, r)$  in (2)-(c)-(ii) is compatible with the fibre structures on  $B_{\text{int}}(p_i, r)$  and  $A_{\text{cap}}(p_i; r_0, r)$  induced from  $f_{i,\text{int}}$  and  $f_{i,\text{cap}}$  respectively.
- (3) There is a singular fibre structure on  $B(p_i, r)$  compatible with the fibre structures on  $B_{\text{int}}(p_i, r)$  and  $A_{\text{cap}}(p_i; r_0, r)$  induced from  $f_{i,\text{int}}$  and  $f_{i,\text{cap}}$  respectively.

First we investigate

*Case A.*  $p \in \partial X$  is a regular boundary point, namely,  $D(\Sigma_p(X))$  is isometric to  $S^1(1)$ .

The argument below provides a parametrized version of collapsing. Namely, when  $(M_i^4, p_i)$  collapses to a product  $(X_0 \times \mathbb{R}, x_0)$ , then collapsing phenomenon can be described as a one-parameter family of collapsing of 3-manifolds. The local version of this is already proved in Section 4.

For sufficiently small  $R \gg r \gg \nu > 0$ , from (12.1)  $\sim$  (12.4) we have a decomposition  $B(p_i, r) = B_{\text{int}}(p_i, r) \cup B_{\text{cap}}(p_i, r)$  and a map  $f_i : B(p_i, r) \rightarrow B(p, r) \cap X_\nu$  such that both  $f_{i,\text{int}}$  and  $f_{i,\text{cap}}$  are fibre bundles. To prove Theorem 12.2 (1)-(a) and (2)-(a), it suffices to show that

**Proposition 12.3.**

$$F_{i,\text{cap}} \simeq \begin{cases} D^3 \text{ or } P^2 \tilde{\times} I & \text{if } F_i = S^2 \\ S^1 \times D^2 \text{ or } K^2 \tilde{\times} I & \text{if } F_i = T^2. \end{cases}$$

Take  $q_i$  with  $d(p_i, q_i) = R$  and  $\varphi_i(q_i) \in \partial X$ , where  $\varphi_i : (M_i, p_i) \rightarrow (X, p)$  is an  $\epsilon_i$ -approximation with  $\lim \epsilon_i = 0$ . Since  $d_{q_i}$ -flow curves are transversal to  $F_{i,\text{cap}}$  and since  $(d_{p_i}, d_{q_i})$  are regular on  $\partial F_{i,\text{cap}}$ ,  $F_{i,\text{cap}}$  is homeomorphic to  $U_r(q_i, p_i) := \partial B(q_i, R) \cap B(p_i, r)$ . Now consider the convergence  $(\frac{1}{\delta_i} M_i, \hat{p}_i) \rightarrow (Y, y_0)$ . Since  $Y$  contains a line,  $Y$  splits isometrically as  $Y = Y_0 \times \mathbb{R}$ . We may think of a soul  $S$  of  $Y$  as a soul of  $Y_0$ . Let  $R_0 \gg R_1 \gg 1$  be sufficiently large, and take  $z_0 \in \{y_0\} \times \mathbb{R}$  with

$d(y_0, z_0) = R_0$  and  $\hat{q}_i \in M_i$  with  $d(\hat{p}_i, \hat{q}_i) = \delta_i R_0$  and  $\hat{q}_i \rightarrow z_0$ . From critical point theory, we obtain  $U_r(q_i, p_i) \simeq U_{\delta_i R_1}(\hat{q}_i, \hat{p}_i)$ , and therefore

$$(12.5) \quad F_{i,\text{cap}} \simeq U_{\delta_i R_1}(\hat{q}_i, \hat{p}_i),$$

$$(12.6) \quad \partial U_{\delta_i R_1}(\hat{q}_i, \hat{p}_i) \simeq F_i.$$

First we consider

*Case A-I.*  $\dim Y = 4$ .

By Theorem 1.6,  $U_{\delta_i R_1}(\hat{q}_i, \hat{p}_i) \simeq U_{R_1}(z_0, y_0)$ . Since  $Y$  has nonnegative curvature,  $U_{R_1}(z_0, y_0)$  is homeomorphic to  $B(y_0, R; Y_0)$ , by which the topology of  $F_{i,\text{cap}}$  is determined.

**Lemma 12.4.** *Suppose that  $p \in \partial X$  is a regular boundary point and  $\dim Y = 4$ .*

- (1) *If  $F_i = S^2$ , then the soul  $S$  is either a point or homeomorphic to  $P^2$ , and Proposition 12.3 holds;*
- (2) *If  $F_i = T^2$ , then  $S$  is isometric to either a circle or a flat Klein bottle and Proposition 12.3 holds.*

*Proof.* If  $\dim S = 0$ , then  $B(y_0, R; Y_0) \simeq D^3$ , which is possible when  $F_i = S^2$ . If  $\dim S = 1$ , then  $B(y_0, R; Y_0) \simeq S^1 \times D^2$ , which is possible when  $F_i = T^2$ . Suppose  $\dim S = 2$ . If  $S$  is orientable, then  $B(y_0, R; Y_0) \simeq S \times I$ , which is impossible because  $\partial F_i$  is connected. If  $S$  is nonorientable, then  $B(y_0, R; Y_0)$  is homeomorphic to a twisted  $I$ -bundle  $S \tilde{\times} I$ , which is possible when  $F_i = S^2$  and  $S \simeq P^2$  and when  $F_i = T^2$  and  $S \simeq K^2$ .  $\square$

*Case A-II.*  $\dim Y = 3$ .

By Theorem 0.2, we have a locally smooth, local  $S^1$ -action  $\psi_i$  on  $B(\hat{p}_i, \delta_i R_1)$  whose orbit space is homeomorphic to  $B(y_0, R_1)$ . Let  $\pi_i : B(\hat{p}_i, \delta_i R_1) \rightarrow B(y_0, R_1)$  be the orbit map. Since  $S_{\delta_3}(Y)$  consists of parallel lines, with the critical point theory, one can construct such a  $\psi_i$  satisfying

$$(12.7) \quad U_{\delta_i R_1}(\hat{q}_i, \hat{p}_i) \simeq \pi_i^{-1}(\{y_0\} \times B(y_0, R_1; Y_0)),$$

(recall the construction of  $\psi_i$  in Sections 6, 7 and 8).

**Lemma 12.5.** *Suppose that  $p \in \partial X$  is a regular boundary point and  $\dim Y = 3$ .*

- (1) *If  $F_i = S^2$ , then  $Y_0$  is homeomorphic to  $\mathbb{R}_+^2$  and  $F_{i,\text{cap}}$  is homeomorphic to either  $D^3$  or  $P^2 \tilde{\times} I$ ;*
- (2) *Suppose  $F_i = T^2$ .*
  - (a) *If  $Y_0$  has no boundary, then either  $Y_0 \simeq \mathbb{R}^2$  and Proposition 12.3 holds, or  $Y_0$  is isometric to a flat Möbius strip and  $F_{i,\text{cap}}$  is homeomorphic to  $K^2 \tilde{\times} I$ ;*

- (b) If  $Y_0$  has nonempty boundary, then  $Y_0$  is isometric to a flat half cylinder and  $F_{i,\text{cap}}$  is homeomorphic to  $S^1 \times D^2$ .

*Proof.* Suppose that  $Y_0$  has no boundary. If  $\dim S = 0$ , then  $Y_0 \simeq R^2$  and the number  $m_i$  of singular orbits of  $\psi_i$  over  $\{y_0\} \times B(y_0, R_1; Y_0)$  is at most two. It follows that  $\pi_i^{-1}(\{y_0\} \times B(y_0, R_1; Y_0))$  is homeomorphic to either  $D^2 \times S^1$  (if  $m_i \leq 1$ ) or  $K^2 \tilde{\times} I$  (if  $m_i = 2$ ), which is possible when  $F_i = T^2$ . If  $\dim S = 1$ ,  $Y_0$  is isometric to either a flat cylinder or a flat Möbius strip. If  $Y_0$  is isometric to a flat cylinder, then  $\pi_i^{-1}(\{y_0\} \times B(y_0, R_1; Y_0))$  is homeomorphic to  $I \times T^2$ , which contradicts (12.6). If  $Y_0$  is isometric to a flat Möbius strip, then  $\pi_i^{-1}(\{y_0\} \times B(y_0, R_1; Y_0))$  is homeomorphic to  $K^2 \tilde{\times} I$ , which is possible when  $F_i = T^2$ .

Suppose that  $\partial Y_0$  is disconnected. Then  $Y_0$  is isometric to a product  $\mathbb{R} \times I$ . It follows that  $\pi_i^{-1}(\{y_0\} \times B(y_0, R_1; Y_0))$  is homeomorphic to  $I \times S^2$ , which contradicts (12.6). Therefore  $\partial Y_0$  is connected. If  $\dim S = 1$ ,  $Y_0$  is isometric to a flat half cylinder. It follows from Theorem 8.1 that  $\pi_i^{-1}(\{y_0\} \times B(y_0, R_1; Y_0))$  is homeomorphic to  $S^1 \times D^2$ , which is possible when  $F_i = T^2$ . If  $\dim S = 0$ ,  $Y_0 \simeq \mathbb{R}_+^2$  and the number  $m_i$  of singular orbits of  $\psi_i$  over  $\{y_0\} \times \text{int } B(y_0, R_1; Y_0)$  is at most one. Hence  $\pi_i^{-1}(\{y_0\} \times B(y_0, R_1; Y_0))$  is homeomorphic to either  $D^3$  (if  $m_i = 0$ ) or  $P^2 \tilde{\times} I$  (if  $m_i = 1$ ).  $\square$

We have just proved Proposition 12.3, and hence Theorem 12.2 (1)-(a) and (2)-(a).

Now consider any singular boundary point  $p \in \partial X$ .

*Case B.*  $p \in \partial X$  is any singular boundary point of  $X$ .

From Theorem 12.2 (1)-(a) and (2)-(a), the topology of  $\partial B(p_i, r)$  is classified as follows: If  $F_i = S^2$ , then

$$(12.8) \quad \partial B(p_i, r) \simeq \begin{cases} S^3 = D^3 \cup D^3 \\ P^3 = D^3 \cup P^2 \tilde{\times} I \\ P^3 \# P^3 = P^2 \tilde{\times} I \cup P^2 \tilde{\times} I, \end{cases}$$

and if  $F_i = T^2$ , then

$$(12.9) \quad \partial B(p_i, r) \simeq \begin{cases} S^1 \times D^2 \cup S^1 \times D^2 \\ S^1 \times D^2 \cup K^2 \tilde{\times} I \\ K^2 \tilde{\times} I \cup K^2 \tilde{\times} I. \end{cases}$$

We determine the topology of  $B(p_i, r)$  under the boundary conditions (12.8) and (12.9) as follows.

**Proposition 12.6.** (1) If  $F_i = S^2$ , then  $B(p_i, r)$  is homeomorphic to either  $D^4$ ,  $S^2 \tilde{\times}_\omega D^2$  ( $|\omega| \in \{1, 2\}$ ), or  $P^2 \tilde{\times}_0 D^2$ ;  
 (2) If  $F_i = T^2$ , then  $B(p_i, r)$  is homeomorphic to either  $D^4$ ,  $S^1 \times D^3$ , or a  $D^2$ -bundle over  $S^2$ ,  $P^2$  or  $K^2$ .

*Proof.* First suppose  $\dim Y = 4$ . If  $F_i = S^2$ , in view of (12.8), we can eliminate the cases when  $\dim S = 1$  or  $S$  is a flat surface, and obtain (1). In the case of  $F_i = T^2$ , it suffices to eliminate the case when  $S$  is isometric to a flat torus. This is done in the following

**Sublemma 12.7.** *If  $p \in \partial X^2$  and  $F_i = T^2$ , then  $B(p_i, r)$  cannot be homeomorphic to a  $D^2$ -bundle over  $T^2$ .*

*Proof.* Suppose that  $B(p_i, r)$  is homeomorphic to a  $D^2$ -bundle over  $T^2$ . Then  $\partial B(p_i, r)$  is an  $S^1$ -bundle over  $T^2$ , and hence  $\Gamma := \pi_1(\partial B(p_i, r))$  is nilpotent. On the other hand, it follows from (12.9) that  $\partial B(p_i, r) \simeq K^2 \tilde{\times} I \cup K^2 \tilde{\times} I$ . By Van Kampen's theorem,  $\Gamma$  is isomorphic to a form  $\Lambda_1 *_{\mathbb{Z}^2} \Lambda_2$ , where  $\Lambda_j \simeq \pi_1(K^2)$  and is considered as the  $\mathbb{Z}_2$ -extension of  $\mathbb{Z}^2$ . Let  $\iota : \Lambda_1 \rightarrow \Lambda_1 * \Lambda_2$  be a natural inclusion and  $\pi : \Lambda_1 * \Lambda_2 \rightarrow \Gamma$  the projection. Then  $\pi \circ \iota : \Lambda_1 \rightarrow \Gamma$  is an injective homomorphism. Since  $\pi_1(K^2)$  is not nilpotent, this is a contradiction.  $\square$

Next suppose  $\dim Y = 3$ . Then in view of (12.8), (12.9) together with  $\dim Y(\infty) \geq 1$ , Theorem 9.1 yields the conclusion.  $\square$

We have just proved Theorem 12.2 (1)-(b) and (2)-(b).

Next we show Theorem 12.2 (1)-(c). Suppose  $B(p_i, r) \simeq P^2 \tilde{\times}_0 D^2$ , and let  $\Gamma_i$  be the deck transformation group of the universal cover  $\pi_i : \tilde{B}(p_i, r) \rightarrow B(p_i, r)$ . Take a sequence  $\mu_i \rightarrow 0$  such that  $(\frac{1}{\mu_i} B(p_i, r), p_i)$  converges to  $(K_p, o_p)$ . We may assume that  $(\frac{1}{\mu_i} \tilde{B}(p_i, r), \tilde{p}_i, \Gamma_i)$  converges to a triplet  $(Z^2, z_0, \Gamma)$ . If  $Z^2$  had nonempty boundary, then we would have a contradiction to Theorem 12.2 (1)-(b) since  $\tilde{B}(p_i, r) \simeq S^2 \times D^2$ . Therefore  $Z^2$  must be isometric to a flat cone without boundary and  $\Gamma \simeq \mathbb{Z}_2$ . It follows from  $Z^2/\Gamma = K_p$  that the action of  $\Gamma$  on  $Z^2$  is by reflection. Let  $\tilde{A}(p_i; r/2, r) := \pi_i^{-1}(A(p_i; r/2, r))$ . By Equivariant Fibration Theorem 18.4, we have a  $\mathbb{Z}_2$ -equivariant  $S^2$ -bundle  $\tilde{A}(p_i; r/2, r) \rightarrow A(z_0; r/2, r)$  and hence a  $\mathbb{Z}_2$ -equivariant homeomorphism  $\tilde{A}(p_i; r/2, r) \simeq S^2 \times A(z_0; r/2, r)$ , where  $\mathbb{Z}_2$  acts diagonally on  $S^2 \times A(z_0; r/2, r)$ ; freely on the  $S^2$ -factor and by reflection on the  $A(z_0; r/2, r)$ -factor. Therefore  $A(p_i; r/2, r)$  is homeomorphic to the diagonal quotient  $(S^2 \times A(z_0; r/2, r))/\mathbb{Z}_2$ . Finally we fill  $B(p_i, r/2)$  with an obvious gluing by  $P^2 \tilde{\times}_0 D^2 \simeq (S^2 \times D^2(r/2))/\mathbb{Z}_2$ . We therefore obtain  $B(p_i, r) \simeq (S^2 \times D^2)/\mathbb{Z}_2$ . This completes the proof of Theorem 12.2 (1)-(c).

In view of Theorem 11.1 (2), the proof of Theorem 12.2(2)-(c) is similar and hence omitted.

Finally we show Theorem 12.2 (3).

**Lemma 12.8.**  *$B(p_i, r)$  is homeomorphic to  $B_{\text{cap}}(p_i, r)$ .*

*Proof.* For a sufficiently small  $\mu \ll \nu$ , take a  $\mu$ -net  $N_\mu$  of  $\partial X \cap B(p, r)$ , and let  $N_{\mu, i}$  be a subset of  $M_i^4$  converging to  $N_\mu$ . Using the flow curves of  $\tilde{d}_{N_{\mu, i}}$ , one can easily prove the lemma.  $\square$

Let us denote the two fibres of  $f_{i,\text{cap}}$  in (12.4) by  $F_{i,\text{cap}}$  and  $F'_{i,\text{cap}}$ . We call the topological types of the triplet  $(B(p_i, r); F_{i,\text{cap}}, F'_{i,\text{cap}})$  the *collapsing data* around  $p_i$ .

Using the topological information on  $B_{\text{cap}}(p_i, r) \simeq B(p_i, r)$  given in (1)-(b) and (2)-(b), we can define a compatible singular fibre structure on  $B_{\text{cap}}(p_i, r)$  by extending the fiber structure on  $\partial B_{\text{cap}}(p_i, r)$  by cone, which is indicated below in terms of the collapsing data.

*Case A.*  $F_i = S^2$ .

(1)

(2)

$$(D^4; D^3, D^3)$$

$$(S^2 \tilde{\times}_{\pm 1} D^2; D^3, D^3)$$

(3)

(4)

$$(S^2 \tilde{\times}_{\pm 2} D^2; D^3, P^2 \tilde{\times} I)$$

$$(P^2 \tilde{\times}_0 D^2; P^2 \tilde{\times} I, P^2 \tilde{\times} I)$$

*Case B.*  $F_i = T^2$ .

(5)

(6)

$$(D^4; S^1 \times D^2, S^1 \times D^2)$$

$$(S^1 \times D^3; S^1 \times D^2, S^1 \times D^2)$$

(7)

(8)

$$(S^1 \times D^3; S^1 \times D^2, K^2 \tilde{\times} I)$$

$$(S^2 \tilde{\times}_{\omega} D^2; S^1 \times D^2, S^1 \times D^2)$$



(9)

(10)

$$(P^2 \tilde{\times}_0 D^2; S^1 \times D^2, K^2 \tilde{\times} I) \quad (\text{a } D^2\text{-bundle over } K^2; K^2 \tilde{\times} I, K^2 \tilde{\times} I)$$

This completes the proof of Theorem 12.2 (3).

In the cases of (1), (4) and (6) in the above list,  $B_{\text{cap}}(p_i, r)$  has a product fibre structure. More precise fiber structures for the cases of (4) and (10) are described in Theorem 12.2(1)-(c) and (2)-(c) respectively. The examples of collapsing related with (5) and (7) are given in Examples 12.10 and 12.12.

To finish the proof of Theorem 12.1, it suffices to show the following

**Lemma 12.9.** *If  $p \in \partial X$  is not an extremal point of  $X$ , then  $B_{\text{cap}}(p_i, r)$  has a product  $F_{i,\text{cap}}$ -fibre structure over  $I$  compatible with those of  $B_{\text{int}}(p_i, r)$  and  $A_{\text{cap}}(p_i; r_0, r)$ .*

*Proof.* Since both  $B_{\text{int}}(p_i, r)$  and  $A_{\text{cap}}(p_i; r_0, r)$  depend on  $\nu$ , we rewrite them as  $B_{\text{int},\nu}(p_i, r)$  and  $A_{\text{cap},\nu}(p_i; r_0, r)$  respectively. We have an obvious homeomorphism

$$A_{\text{cap},\nu}(p_i; r_0, r) - \text{int } A_{\text{cap},\nu/2}(p_i; r_0, r) \simeq (F_i \times I \times \partial I) \times I.$$

In a way similar to Lemma 4.11, we have

$$(12.10) \quad A_{\text{cap},\nu/2}(p_i; r_0, r) \simeq F_{i,\text{cap}} \times I \times \partial I,$$

together with a flow  $\phi$  giving the above product structure (12.10) on  $A_{\text{cap},\nu/2}(p_i; r_0, r)$  (compare also the argument in Section 8). This provides a trivial  $F_i$ -bundle  $\xi_1$  on  $\partial A_{\text{cap},\nu/2}(p_i; r_0, r) \simeq F_i \times I \times \partial I \times \{1\}$ . We also have the trivial  $F_i$ -bundle  $\xi_0$  on  $\partial A_{\text{cap},\nu}(p_i; r_0, r) \simeq F_i \times I \times \partial I \times \{0\}$ . For any  $(x, u) \in F_i \times I \times \partial I \times \{1\}$ , let  $t(x, u) \in \mathbb{R}$  denote the parameter at which the flow curve  $\phi_{(x,u)}$  starting from  $(x, u)$  meets the fibre  $F(\xi_1)_u$  of  $\xi_1$  over  $u$ . Note that the fibres  $F(\xi_1)_u$  are not compatible with the fibres  $\partial F_{i,\text{cap}}$ . Then  $\varphi_s(x, u) := \phi_{(x,u)}(st(x, u))$ ,  $0 \leq s \leq 1$ , presents a one-parameter family of homeomorphisms  $\varphi_s$  of  $F_i \times I \times \partial I$ . Define a homeomorphism  $\Phi$  of  $F \times I \times \partial I \times I$  by  $\Phi(x, u, s) := (\varphi_s(x, u), s)$ . This gives a one-parameter family of trivial  $F_i$ -bundle joining  $\xi_0$  and  $\xi_1$ , which defines the required product  $F_{i,\text{cap}}$ -fibre structure on  $B_{\text{cap}}(p_i, r)$ .  $\square$

**Example 12.10.** Let  $T^2 = SO(2) \times SO(2)$  act on  $D^2 \times D^2$  by

$$(e^{i\varphi}, e^{i\theta}) \cdot (r_1 e^{i\alpha_1}, r_2 e^{i\alpha_2}) = (r_1 e^{i(\alpha_1 + m_1 \varphi + n_1 \theta)}, r_2 e^{i(\alpha_2 + m_2 \varphi + n_2 \theta)}),$$

where we assume

$$\begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} = \pm 1$$

to make the action effective. Then we have a sequence of metrics  $g_i$  on  $D^2 \times D^2$  of nonnegative sectional curvature such that  $(D^2 \times D^2, g_i)$  collapses to  $I \times I$ . In this case, we have the same collapsing data as (5).

**Example 12.11.** By gluing  $T^2$ -equivariantly suitable two  $T^2$ -action on  $D^2 \times D^2$  as described in Example 12.10, one can construct an  $T^2$ -action on  $S^2 \tilde{\times}_\omega D^2$  for any  $\omega$  with the orbit space homeomorphic  $I^2$  having two fixed points on  $\partial I^2$  (see [16]).

To be more explicit, let us restrict our attention to the case  $\omega = 1$ . Let  $E$  be an  $\mathbb{R}^2$ -vector bundle over  $S^2$  with a fiber metric such that the unit disk-bundle  $E_1 = S^2 \tilde{\times}_1 D^2(1) \subset E$  has boundary homeomorphic to  $S^3$ . For any  $r > 0$ , we have a faithful  $T^2$ -action on  $\partial(S^2 \tilde{\times} D^2(r))$ . This induces the  $T^2$ -action on  $E$  with  $E/T^2 \simeq I \times [0, \infty)$ . For a  $T^2$ -invariant metric  $g$  on  $E_1$ , we have a sequence of metrics  $g_i$  on  $E_1$  such that  $(E_1, g_i)$  collapses to  $(E_1/T^2, \bar{g})$ . In this case for any extremal point  $p \in \partial I \times [0, \infty)$ ,  $B(p_i, r) \simeq D^4$  with the same collapsing data as (5).

**Example 12.12.** In the product  $D^2 \times \mathbb{R}^2$ , identify  $(x, t)$  and  $(x^*, t')$  for any  $x \in \partial D^2$  and  $t \in \mathbb{R}^2$ , where  $x^*$  and  $t'$  denote the antipodal point of  $x^*$  and the image of  $t$  by a reflection respectively. We consider the resulting identification space  $D^2 \times \mathbb{R}^2 / \sim$ , which is an  $\mathbb{R}^2$ -bundle over  $P^2$  denoted  $P^2 \tilde{\times}_0 \mathbb{R}^2$ . Let  $g_i$  and  $h_i$  be sequences of rotationally symmetric nonnegatively curved metrics on  $D^2$  and  $\mathbb{R}^2$  such that  $(D^2, g_i)$  and  $(\mathbb{R}^2, o, h_i)$  converge to  $I$  and  $[0, \infty)$  respectively. Then the product  $(D^2 \times \mathbb{R}^2, g_i \times h_i)$  gives a nonnegatively curved metric  $k_i$  on  $P^2 \tilde{\times}_0 \mathbb{R}^2$  such that  $(P^2 \tilde{\times}_0 \mathbb{R}^2, o, k_i)$  collapses to the product  $I \times [0, \infty)$ . For the extremal point  $p \in \partial(I \times [0, \infty))$  corresponding to  $\partial D^2 \times 0$ ,  $B(p_i, r)$  is homeomorphic to a  $D^2$ -bundle over Mö, that is  $S^1 \times D^3$ , and has the same collapsing data as (7).

**Problem 12.13.** Determine if there exists a sequence of collapsed metrics on a  $D^2$ -bundle over  $S^2$  or  $P^2$  with a definite lower sectional curvature bound having the same collapsing data as (2), (3), (8) or (9).

### 13. CLASSIFICATION OF COLLAPSING TO NONCOMPACT TWO-SPACES WITH NONNEGATIVE CURVATURE

Let a sequence of pointed complete 4-dimensional orientable Riemannian manifolds  $(M_i^4, p_i)$  with  $K \geq -1$  collapses to a pointed complete noncompact 2-dimensional Alexandrov space  $(Y^2, y_0)$  with nonnegative curvature. In this section, using the results of Sections 10, 11 and 12, we classify the topology of a large metric ball  $B(p_i, R)$  in terms of geometric properties of  $Y^2$ . The classification result will be used in the next section to describe the phenomena of orientable 4-manifolds collapsing to a closed interval.

Applying Theorems 0.6 and 12.1 to the convergence  $(M_i^4, p_i) \rightarrow (Y^2, y_0)$ , we have a singular fibration  $\pi_i : B(p_i, R) \rightarrow B(y_0, R)$  with

general fibre  $F_i$ , either  $S^2$  or  $T^2$ , where  $R$  is a large positive number. Actually, we have such a singular fibre structure on a small perturbation of  $B(p_i, R)$ , which is homeomorphic to  $B(p_i, R)$ . Let  $m_i$  and  $n_i$  denote the numbers of singular fibres of  $\pi_i : B(p_i, R) \rightarrow B(y_0, R)$  over  $B(y_0, R) \cap \text{int } Y$  and over  $B(y_0, R) \cap \partial Y$  respectively. From the classification of Alexandrov surfaces with nonnegative curvature (cf. [43]) together with Theorems 0.6 and 12.1, if  $Y$  has no boundary, then  $m_i \leq 2$ , and if  $Y$  has nonempty boundary, then  $m_i \leq 1$ ,  $n_i \leq 2$  and the possible cases are

$$(m_i, n_i) = (0, 0), \quad (0, 1), \quad (0, 2), \quad (1, 0),$$

in the latter case. Note also that  $m_i = 0$  if  $F_i = S^2$ .

**Theorem 13.1.** *Under the situation above, the topology of  $B(p_i, R)$  can be classified as follows:*

Case I.  $\dim Y(\infty) = 1$ .

- (1) *Suppose  $Y$  has no boundary.*
  - (a) *If  $F_i = S^2$ ,  $B(p_i, R) \simeq D^2 \times S^2$ ;*
  - (b) *If  $F_i = T^2$ ,  $B(p_i, R) \simeq D^2 \times T^2$ .*
- (2) *Suppose  $Y$  has nonempty boundary.*
  - (a) *If  $F_i = S^2$ ,  $B(p_i, R)$  is homeomorphic to either  $D^4$ ,  $S^2 \tilde{\times}_\omega D^2$  with  $|\omega| \in \{1, 2\}$  or  $P^2 \tilde{\times}_0 D^2$ ;*
  - (b) *If  $F_i = T^2$ ,  $B(p_i, R)$  is homeomorphic to either  $D^4$ ,  $S^1 \times D^3$ , or a  $D^2$ -bundle over  $S^2$ ,  $P^2$  or  $K^2$ .*

Case II.  $\dim Y(\infty) = 0$ .

- (1) *Suppose  $Y$  has no boundary.*
  - (i) *If  $Y \simeq \mathbb{R}^2$ , then the following holds :*
    - (a) *If  $F_i = S^2$ ,  $B(p_i, R)$  is homeomorphic to the space in Case I-(1)-(a) ;*
    - (b) *If  $F_i = T^2$ ,  $B(p_i, R)$  is homeomorphic to either the space in Case I-(1)-(b), or an  $I$ -bundle over  $T^2 \tilde{\times} S^1$ , a twisted  $S^1$ -bundle over  $T^2$  doubly covered by  $T^3$ .*
  - (ii) *If  $Y$  is isometric to either a flat cylinder or a flat Möbius strip, then  $B(p_i, R)$  is homeomorphic to an  $I$ -bundle over an  $F_i$ -bundle over  $S^1$ .*
- (2) *Suppose  $Y$  has nonempty boundary.*
  - (i) *If  $Y \simeq \mathbb{R}_+^2$ , then the following holds:*
    - (a) *If  $(m_i, n_i) = (0, 2)$ , then  $B(p_i, R)$  is homeomorphic to a gluing of two disk-bundles, denoted  $N_j^{k_j} \tilde{\times} D^{4-k_j}$ ,  $j = 1, 2$ , over  $k_j$ -dimensional closed manifolds  $N_j^{k_j}$  with nonnegative Euler numbers, where  $0 \leq k_j \leq 2$ . The gluing is done along one of  $D^3$ ,  $P^2 \tilde{\times} I$  (if*

- $F_i = S^2$ ), and one of  $S^1 \times D^2$ ,  $K^2 \tilde{\times} I$  (if  $F_i = T^2$ ), imbedded in the boundaries of  $N_j^{k_j} \tilde{\times} D^{4-k_j}$ ;
- (b) If  $(m_i, n_i) = (1, 0)$ , then  $B(p_i, R)$  is homeomorphic to one of the gluings:

$$S^1 \times D^3 \bigcup_{T^2 \times I} T^2 \times D^2, \quad K^2 \tilde{\times} D^2 \bigcup_{T^2 \times I} T^2 \times D^2.$$

- (ii) If  $Y$  is isometric to a flat half cylinder, then the following holds:
- (a) If  $F_i = S^2$ ,  $B(p_i, R)$  is homeomorphic to either  $S^1 \times D^3$  or an  $I$ -bundle over a  $S^1 \times P^2$ ;
- (b) If  $F_i = T^2$ ,  $B(p_i, R)$  is homeomorphic to either  $D^2$ -bundle over  $T^2$  or  $K^2$ , or an  $I$ -bundle over a  $K^2$ -bundle over  $S^1$ .
- (iii) If  $Y$  is isometric to a product  $I \times \mathbb{R}$ , then the following holds:
- (a) If  $F_i = S^2$ ,  $B(p_i, R)$  is homeomorphic to either  $S^3 \times I$ ,  $P^3 \times I$  or  $(P^3 \# P^3) \times I$ ;
- (b) If  $F_i = T^2$ ,  $B(p_i, R)$  is homeomorphic to a form  $(U^3 \cup V^3) \times I$ , where  $U^3$  and  $V^3$  are ones of  $S^1 \times D^2$  and  $K^2 \tilde{\times} I$ , and  $U^3 \cup V^3$  denotes a gluing along their boundary tori.

*Proof.* In Case I-(1),  $Y$  is homeomorphic to  $\mathbb{R}^2$  and has at most one essential singular point. In Case I-(2), there are no essential singular points in  $\text{int } Y$  and at most one extremal point on  $\partial Y$ . Hence the conclusions are the direct consequences of Theorems 0.6 and 12.1.

In Case II-(1)-(i), in view of Case I-(1) and Theorem 0.6 we may assume that  $F_i = T^2$  and  $m_i = 2$ . Therefore  $Y$  is isometric to the double  $D(I \times [0, \infty))$  for some closed interval  $I$ . Split  $B(y_0, R)$  with a proper segment into two closed domain  $B_1$  and  $B_2$  each of which contains one of the two essential singular points of  $Y$ . Applying Theorem 11.1 to each  $B_j$ , we have

$$B(p_i, R) \simeq T^2 \times D^2 \bigcup_{I \times T^2} T^2 \times D^2.$$

But actually  $\text{int } B(p_i, R)$  is homeomorphic to the complete flat manifold defined as the  $\mathbb{Z}_2$ -quotient of  $T^2 \times S^1 \times \mathbb{R}$  by an involution, where the action of  $\mathbb{Z}_2$  is diagonal, free on  $T^2$ , and orientation reversing on both factors  $S^1$  and  $\mathbb{R}$ . Therefore  $B(p_i, R) \simeq ((T^2 \times S^1)/\mathbb{Z}_2) \tilde{\times} I$ .

Consider Case II-(1)-(ii), the case of  $\dim S = 1$ . Obviously  $B(p_i, R)$  is an  $F_i$ -bundle over  $B(y_0, R)$ , and the conclusion follows.

Next consider Case II-(2)-(i). If  $(m_i, n_i) = (0, 2)$ , then  $Y$  is isometric to  $I \times [0, \infty)$  for some closed interval  $I$ . Split  $B(y_0, R)$  with a proper segment into two closed regions  $B_1$  and  $B_2$  each of which contains one of the two extremal points in  $\partial Y$ . Applying Theorem 12.1 to each  $B_j$ ,

we obtain the required gluing. If  $(m_i, n_i) = (1, 0)$ , then  $Y$  is isometric to  $D([0, \infty) \times \{x \geq 0\}) \cap \{x \leq a\}$  for some  $a > 0$ . By a similar cutting and pasting argument, we obtain the required gluing.

Consider the cases II-(2)-(ii) and (iii), the cases of  $\dim S = 1$  and  $\partial Y$  being disconnected respectively. By Theorem 12.1 together with the facts that the mapping class group  $\mathcal{M}(P^2)$  of all homeomorphisms of  $P^2$  is trivial and  $\mathcal{M}_+(K^2 \tilde{\times} I) = \mathcal{M}(K^2)$ , we obtain the conclusions.  $\square$

We are in a position to prove Theorem 0.9. Let us consider a sequence of 4-dimensional closed orientable Riemannian manifolds  $M_i^4$  with  $K \geq -1$  converging to a 1-dimensional closed interval  $X^1$ . Let  $\{p, q\} := \partial X$  and take  $p_i$  and  $q_i$  in  $M_i^4$  with  $p_i \rightarrow p$  and  $q_i \rightarrow q$ . By Fibration Theorem 1.2,  $M_i^4$  is homeomorphic to a gluing

$$B(p_i, r) \bigcup B(q_i, r),$$

for any sufficiently small positive number  $r$  and any large  $i$  compared to  $r$ , where  $\partial B(p_i, r) \simeq \partial B(q_i, r)$  is homeomorphic to one of the closed 3-manifolds given in Theorem 0.8. Now we consider the local convergence  $B(p_i, r) \rightarrow B(p, r)$ . By Theorem 4.1, we have sequences  $\delta_i \rightarrow 0$  and  $\hat{p}_i \rightarrow p$  such that

- (1) for any limit  $(Y, y_0)$  of  $(\frac{1}{\delta_i} M_i, \hat{p}_i)$ , we have  $\dim Y \geq 2$ ;
- (2)  $B(p_i, r)$  is homeomorphic to  $B(\hat{p}_i, R\delta_i)$  for every  $R \geq 1$  and large  $i$  compared to  $R$ .

Applying Theorems 9.1 and 13.1 to the new convergence  $(\frac{1}{\delta_i} M_i, \hat{p}_i) \rightarrow (Y, y_0)$  together with the boundary condition stated above, we can determine or classify the possible topological type of  $(p_i, r)$  as:

**Theorem 13.2.** *If  $p \in \partial X^1$ , then  $B(p_i, r)$  is homeomorphic to a disk bundle over a  $k$ -dimensional closed manifold  $N^k$  with  $0 \leq k \leq 3$ , or a gluing of two disk-bundles over  $k_j$ -dimensional closed manifolds  $Q^{k_j}$ ,  $j = 1, 2$ , with  $0 \leq k_j \leq 2$ , where  $N^k$  and  $Q^{k_j}$  have nonnegative Euler numbers, and if  $k = 3$ ,  $N^3$  is one of the closed 3-manifolds given in Theorem 0.8.*

We have a similar topological information on  $B(q_i, r)$ . This completes the proof of Theorem 0.9.

In the situation of Theorem 0.9, the author does not know a specific example of a closed 4-manifold built of gluing of three or four pieces of disk-bundles which admits a sequence of metrics collapsing to a closed interval under  $K \geq -1$ . In view of Proposition 9.15, one of typical problems arising from the results of the present paper is the following:

**Problem 13.3.** Letting  $M^4$  be either  $S^2 \times S^2 \# S^2 \times S^2$  or the connected sum of three or four pieces of  $\pm \mathbb{C}P^2$ , determine if  $M^4$  admits a sequence of metrics collapsing to a closed interval under  $K \geq -1$ .

## Part 2. Complete Alexandrov spaces with nonnegative curvature

In Part 2, we develop the geometry of complete Alexandrov spaces with nonnegative curvature. Most of them are needed in Part 1. In particular, we establish the Generalized Soul Theorem 2.6.

Throughout this part, let  $X$  be an  $n$ -dimensional complete Alexandrov space with nonnegative curvature. We assume  $X$  to be either noncompact or having nonempty boundary. Applying the Cheeger-Gromoll basic construction, we obtain a sequence of finitely many nonempty compact totally convex sets of  $X$ :

$$(13.1) \quad C(0) \supset C(1) \supset C(2) \supset \cdots \supset C(k),$$

with  $S = C(k)$  as in Section 2 except that  $C(0)$  coincides with the minimum set of  $d(\partial X, \cdot)$  if  $X$  is compact.

From the construction, we have the filtration  $\{X^t\}_{0 \leq t < t_*}$ ,  $t_* \leq \infty$ , by compact totally convex subsets such that

- (1)  $X^s = \{x \in X^t \mid d(x, \partial X^t) \geq t - s\}$  for  $s \leq t$ ;
- (2) the closure of  $\bigcup_{0 \leq t < t_*} X^t$  coincides with  $X$ ;
- (3)  $X^0 = C$ .

Note  $t_* < \infty$  if and only if  $X$  is compact.

### 14. LOCAL REGULARITIES

In this section, we are concerned with three local regularity properties of  $X$ .

We begin with a more general situation described as follows: Let  $X$  be an  $n$ -dimensional complete nonnegatively curved Alexandrov space, and suppose  $X$  has a filtration  $\{X_*^t\}_{0 \leq t < t_*}$  by closed totally convex subsets satisfying the same conditions as (1), (2), (3) above, which is determined by a convex function  $\rho$  with minimum 0 in such a way that  $X_*^t = \{\rho \leq t\}$ , where  $t_*$  is the supremum of the values of  $\rho$ . However we do not assume that each  $X_*^t$  is compact here. Put  $C_* := X_*^0$  and let us assume  $\dim C_* = n - 1$ .

A point  $x \in C_*$  is called a *one-normal point* (resp. *two-normal point*) if there exists exactly one (resp. two) geodesic ray(s) from  $x$  perpendicular to  $C_*$ . Note that each point of  $C_*$  is either a one-normal point or a two-normal point (see [43]).

**Proposition 14.1.** *Under the hypothesis above, suppose that  $p \in \text{int } C_*$  is a topologically nice point of  $X^n$ . Then*

- (1)  $\Sigma_p(X)$  is isometric to the spherical suspension over  $\Sigma_p(C_*)$ ;
- (2)  $p$  is a two-normal point.

*Proof.* In view of Proposition 2.1, it suffices to show only (2). This is done by induction on  $n$ . Under the convergence  $(\frac{1}{r}X, p) \rightarrow (K_p, o_p)$  as  $r \rightarrow 0$ , the filtration  $\{X_*^t\}_{0 \leq t < t_*}$  gives rise to a filtration  $\{K_p^t\}_{0 \leq t < \infty}$

with  $K_p^0 = K_p(C_*)$  satisfying the same conditions as (1), (2), (3) above. This is done as follows. Let  $\rho_\infty$  be the limit of  $m\rho$  under the convergence  $(mX, p) \rightarrow (K_p, o_p)$ ,  $m \rightarrow \infty$ . Then  $K_p^t = \{\rho_\infty \leq t\}$ . By Proposition 2.1,  $K_p^0 = K_p(C_*)$ . Obviously

$$\Sigma_p(C_*) = \Sigma_p(C_*) \times 1 \subset K_p(C_*) \subset K_p(X).$$

We show that any point  $v$  of  $\Sigma_p(C_*)$  is a two-normal point of  $K_p(C_*)$ . Note that  $K_v(K_p X) = \mathbb{R} \times K_v(\Sigma_p X)$  and that  $K_v(\Sigma_p X)$  has a filtration  $\{K_v^t\}_{0 \leq t < \infty}$  with  $K_v^0 = K_v(\Sigma_p C_*)$  satisfying the conditions as above. From the assumption,  $\Sigma_v(\Sigma_p X) \simeq S^{n-2}$  and  $v$  is a topologically nice point of  $\Sigma_p(X)$ . Thus we can apply the induction hypothesis to the filtration  $\{K_v^t\}_{0 \leq t < \infty}$  to conclude that  $v = o_v$  is a two-normal point of  $K_v(\Sigma_p C_*)$  and therefore of  $K_p(C_*)$ .

Now suppose that  $o_p$  is a one-normal point of  $K_p(C_*)$  with a unique direction  $\xi \in \Sigma_p(X)$  normal to  $\Sigma_p(C_*)$ . From the previous argument together with Proposition 2.1,  $\Sigma_\xi(\Sigma_p X)$  is a double cover of  $\Sigma_p(C_*)$ . In particular  $\pi_1(\Sigma_p(C_*)) = \mathbb{Z}_2$ . Furthermore  $\Sigma_p(C_*)$  is a deformation retract of  $\Sigma_p(X) - \text{int } B(\xi, \epsilon)$  for a small  $\epsilon > 0$ . However by the assumption,  $\Sigma_p(X) - \text{int } B(\xi, \epsilon) \simeq D^{n-1}$ , a contradiction.  $\square$

For a subset  $D \subset C_*$ ,  $\mathcal{N}(D)$  denotes the union of all the geodesic rays starting from the points of  $D$  perpendicularly to  $C_*$ . By Propositions 2.1 and 14.1, if  $D \subset \text{int } C_*$  and if any point of  $D$  is topologically nice in  $X$ , then  $\mathcal{N}(D)$  is a line-bundle over  $D$ .

Now we go back to our situation in (13.1). For simplicity we put  $C := C(0)$ .

Let us first assume that  $\partial C$  is nonempty. For a small  $\epsilon > 0$ , consider the following function

$$f_\epsilon(x) = d(C_\epsilon, x)$$

on  $X - C_\epsilon$ , where

$$C_\epsilon = \{x \in C \mid d(\partial C, x) \geq \epsilon\}.$$

The local regularity property we next study is related with the regularity of  $f_\epsilon$ . Note that the critical point set of  $f_\epsilon$  is contained in  $\partial C$ .

Let  $p \in \partial C$  be given. In view of the convergence  $(\frac{1}{\delta}X, p) \rightarrow (K_p, o_p)$ , the following lemma is obvious.

**Lemma 14.2.** *For any  $p \in \partial C$  there exist positive numbers  $\epsilon_p$ ,  $\delta_p$  and  $c_p > 1$  such that*

$$B(p, \delta) \cap \partial C \subset \{\epsilon/2 \leq f_\epsilon \leq c_p \epsilon\}$$

for every  $\delta \leq \delta_p$  and  $\epsilon$  with  $\epsilon/\delta \leq \epsilon_p$ .

From the filtration  $\{X^t\}_{t \geq 0}$ , we obtain the filtration  $\{K_p^t\}_{t \geq 0}$ , satisfying the same conditions as (1), (2), (3) in the beginning of Part 2, of

$K_p$  by totally convex sets, in a way similar to the proof of Proposition 14.1. Let  $C_\infty$  be the limit of  $C$  under the above convergence and

$$C_{\infty\epsilon} = \{x \in C_\infty \mid d(\partial C_\infty, x) \geq \epsilon\},$$

which is the limit of  $mC_{\epsilon/m}$ . Note that  $C_\infty$  does not necessary coincide with  $K_p^0$  because of  $p \in \partial C$ .

We shall also consider the function

$$f_{\infty\epsilon}(x) = d(C_{\infty\epsilon}, x),$$

on  $K_p - C_{\infty\epsilon}$ , where

$$C_{\infty\epsilon} = \{x \in C_\infty \mid d(\partial C_\infty, x) \geq \epsilon\}.$$

**Lemma 14.3.** *Suppose  $\dim X = 4$  and  $\dim C \in \{2, 3\}$ . Then for any  $p \in \partial C$  and  $c \geq c_p$ , there exist positive numbers  $\epsilon_{p,c}$ ,  $\delta_{p,c}$  and  $\mu_p$  such that for every  $\delta' < \delta \leq \delta_{p,c}$  and  $\epsilon$  with  $\epsilon/\delta' \leq \epsilon_{p,c}$*

- (1)  $(f_\epsilon, d_p)$  is regular on  $\{\delta' \leq d_p \leq \delta, \epsilon/2 \leq f_\epsilon \leq c\epsilon\} - B(\partial C, \epsilon/100)$ ;
- (2)  $(f_\epsilon, d_C)$  is regular on  $\{d_p \leq \delta, \epsilon/3 \leq f_\epsilon \leq 2\epsilon/3, 0 < d_C \leq \mu_p\epsilon\}$ ;
- (3)  $(f_\epsilon, d_p, d_C)$  is regular on  $\{\delta' \leq d_p \leq \delta, \epsilon/3 \leq f_\epsilon \leq 2\epsilon/3, 0 < d_C \leq \mu_p\epsilon\}$ .

*Proof.* Under the convergence  $(\frac{1}{\delta}X, p) \rightarrow (K_p, o_p)$ ,  $C_{\delta\epsilon}$ ,  $f_{\delta\epsilon}$ ,  $d_p$  and  $d_C$  converge to  $C_{\infty\epsilon}$ ,  $f_{\infty\epsilon}$ ,  $d_{o_p}$  and  $d_{C_\infty}$  respectively. Therefore it suffices to show that

- (1)'  $(f_{\infty\epsilon}, d_{o_p})$  is regular on  $\{\delta' \leq d_{o_p} \leq 1, \epsilon/2 \leq f_{\infty\epsilon} \leq c\epsilon\} - B(\partial C_\infty, \epsilon/100)$ ;
- (2)'  $(f_{\infty\epsilon}, d_{C_\infty})$  is regular on  $\{d_{o_p} \leq 1, \epsilon/3 \leq f_{\infty\epsilon} \leq 2\epsilon/3, 0 < d_C \leq \mu_p\epsilon\}$ ;
- (3)'  $(f_{\infty\epsilon}, d_{o_p}, d_{C_\infty})$  is regular on  $\{\delta' \leq d_{o_p} \leq 1, \epsilon/3 \leq f_{\infty\epsilon} \leq 2\epsilon/3, 0 < d_{C_\infty} \leq \mu_p\epsilon\}$ .

First suppose  $\dim C = 3$ . We show (1)'. For every  $x \in \{\delta' \leq d_{o_p} \leq 1, \epsilon/2 \leq f_{\infty\epsilon} \leq c\epsilon\} - B(\partial C_\infty, \epsilon/100)$ , let  $y \in \partial C_{\infty\epsilon}$ ,  $z \in \partial C_\infty$  and  $u \in C_\infty$  be nearest points of  $\partial C_{\infty\epsilon}$ , of  $\partial C_\infty$  and of  $C_\infty$  respectively from  $x$ . Note that

$$|\tilde{Z}_{o_p}xy - \pi/2| < \tau(\delta', \epsilon/\delta').$$

We must show that

$$\tilde{Z}_{o_p}xw > \pi/2 + c, \quad \tilde{Z}yxw > \pi/2 + c,$$

for some point  $w$ , where  $c$  is a uniform positive constant not depending on  $\epsilon$ . Letting  $a$  be a point on the ray from  $o_p$  through  $x$  with  $d(o_p, a) > d(o_p, x)$ , we obtain  $\tilde{Z}yxa > \pi/2 - \tau(\delta', \epsilon/\delta')$ .

We consider the following three cases.

*Case 1.*  $d(z, u) \geq \epsilon/1000$ .

Let  $b \in \partial C_\infty$  and  $v \in C_\infty$  be such that  $d(o_p, b) = d(o_p, z) + d(z, b)$  and  $zbvu$  forms a square in  $C_\infty$ . Now observe that the normal bundle



$\mathcal{N}(\text{int } C_\infty)$  over  $\text{int } C_\infty$  is naturally imbedded in  $K_p$ . Let  $z_1, b_1$  and  $v_1$  be points in  $\mathcal{N}(\text{int } C_\infty)$  such that  $uzbvzx_1b_1v_1$  forms a parallelepiped in  $\mathcal{N}(\text{int } C_\infty)$ . Then we have

$$\tilde{\angle} yxb_1 > \pi/2 + c_1, \quad \tilde{\angle} o_pxb_1 > \pi/2 + c_1,$$

for some uniform constant  $c_1 > 0$ . This implies that  $(f_\epsilon, d_{o_p})$  is  $(c_1, \tau(\delta', \epsilon/\delta'))$ -regular at  $x$ .

*Case 2.*  $d(z, u) \leq \epsilon/1000$  and  $\tilde{\angle} xyz \geq 1/100$ .

Let  $x_1$  be the point on  $xy \cap \mathcal{N}(C_\infty)$  such that  $\tilde{\angle} yux_1 = \pi/2$  ( $x_1 = x$  if  $z \neq u$ ). Take  $v \in \mathcal{N}(C_\infty)$  such that  $d(u, v) = d(u, x_1) + d(x_1, v)$  and  $d(x, x_1)/d(v, x)$  is sufficiently small. Then  $\tilde{\angle} yxv > \pi/2 + c_2$ . Let  $w$  be a point such that  $w'_x$  is a midpoint between  $v'_x$  and  $a'_x$ . Then we have

$$\tilde{\angle} zxw > \pi/2 + c'_2, \quad \tilde{\angle} o_p xw > \pi/2 + c'_2,$$

for some uniform constant  $c'_2 > 0$ . This implies that  $(f_\epsilon, d_{o_p})$  is  $(c, \tau(\delta', \epsilon/\delta'))$ -regular at  $x$ .

*Case 3.*  $d(z, u) \leq \epsilon/1000$  and  $\tilde{\angle} xyz \leq 1/100$ .

Note that  $z = u$  in this case. Let  $\delta_1$  be a small positive number, and let  $R_{\delta_1}$  denote the set of  $(4, \delta_1)$ -strained points in  $\partial C_\infty \cap B(o_p, 1)$ . By Lemma 1.8 of [47], we have the following sublemma.

**Sublemma 14.4.** *There is a small  $\epsilon > 0$  so that for every  $x \in B(R_{\delta_1}, \epsilon)$  and  $z \in \partial C_\infty$  with  $d(x, z) = d(x, \partial C_\infty)$ , there exists a point  $v$  satisfying  $\tilde{\angle} zxv > \pi - \delta_1$ .*

Now suppose that the required regularity property does not hold in Case 3. Then we have a sequence  $\epsilon_i$  of positive numbers tending to 0 and a sequence  $x_i \in \{\epsilon_i/2 \leq f_{\infty \epsilon_i} \leq c\epsilon_i, \delta' \leq d_{o_p} \leq 1\} - B(\partial C_\infty, \epsilon_i/100)$  such that

$$\tilde{\angle} o_px_iw \leq \pi/2 + o_i, \quad \text{or} \quad \tilde{\angle} y_ix_iw \leq \pi/2 + o_i$$

for any point  $w$ , where  $y_i \in (C_\infty)_{\epsilon_i}$  and  $z_i \in \partial C_\infty$  denote nearest points of  $(C_\infty)_{\epsilon_i}$  and of  $C_\infty$  from  $x_i$  respectively, and  $\lim o_i = 0$ . Passing to a subsequence, we may assume that  $(\frac{1}{\epsilon_i}K_p, x_i)$  converges to a pointed nonnegatively curved Alexandrov space  $(\hat{X}, x_\infty)$ . There exists a filtration  $\{\hat{X}^t\}_{t \geq 0}$  of  $\hat{X}$  by totally convex subsets induced from  $\{K_p^t\}_{t \geq 0}$  as before. Note that  $\hat{X}^0$  is not necessary the limit of  $C_\infty$ . Let  $o_\infty \in \hat{X}(\infty)$ ,  $y_\infty \in \hat{X}$  and  $z_\infty \in \hat{X}$  are the limits of  $o_p, y_i$  and  $z_i$  respectively. If  $d(x_\infty, \hat{X}^0) > 0$ , letting  $t_0$  be such that  $x_\infty \in \partial \hat{X}^{t_0}$ , we can find a point  $w \in \hat{X} - \hat{X}^{t_0}$  such that

$$\tilde{\angle} o_\infty x_\infty w \geq \pi/2, \quad \tilde{\angle} y_\infty x_\infty w \geq \pi/2 + c,$$

for some  $c > 0$ . Now it is possible to take  $w_1$  near  $w$  such that

$$\tilde{\angle} o_\infty x_\infty w_1 \geq \pi/2 + c_1, \quad \tilde{\angle} z_\infty x_\infty w_1 \geq \pi/2 + c_1,$$

for some  $c_1 > 0$ . This yields a contradiction.

Thus we may assume that  $x_\infty \in \hat{X}^0$ . Since this implies  $z_\infty \in \text{int } \hat{X}^0$ , there are exactly two directions at  $z_\infty$  normal to  $\hat{X}^0$  (see Proposition 14.1). Since we may assume that  $\lim \tilde{\angle} x_i z_i y_i = \pi$ , it follows that  $z_\infty$  is a  $(4, 0)$ -strained point. Therefore  $z_i \in R_{\delta_1}$  for large  $i$ . From the choice of  $\epsilon$ , it is possible to take  $v_i$  satisfying  $\tilde{\angle} z_i x_i v_i > \pi/2 + c_3$  for large  $i$ . Obviously  $\tilde{\angle} y_i x_i v_i > \pi/2 + c_3/2$ . Taking  $w_i$  as in Case 2, we would obtain the regularity of  $(f_\epsilon, d_{op})$  at  $x_i$ , a contradiction.

(2)' follows from the existence of  $\mathcal{N}(C_\infty)$ . (3)' follows from (2)' and the idea of the proof of (1)'. Therefore the details are omitted.

Next suppose  $\dim C = 2$ . In this case, Sublemma 14.4 obviously holds, and similar arguments applies. Therefore the details are omitted.  $\square$

The last local regularity property is a purely metric one on the tangent cone. The following Proposition is also important in the proof of Theorem 2.6.

**Proposition 14.5.** *Let  $X^4$  be a 4-dimensional complete Alexandrov space with nonnegative curvature, and  $C(i)$  as in (13.1),  $0 \leq i \leq k$ . If  $p \in \text{int } C(i)$  is a topological regular point of  $X^4$ , then  $K_p(X^4)$  is isometric to the product of  $K_p(\text{int } C(i))$  and a Euclidean cone.*

Proposition 14.5 is true for the 3-dimensional complete open Alexandrov spaces with nonnegative curvature (see [43]).

**Conjecture 14.6.** Proposition 14.5 is true for every complete Alexandrov space with nonnegative curvature if  $p \in \text{int } C(i)$  is a topological regular point of  $X^n$ .

*Remark 14.7.* Let  $X$  be the Euclidean cone over the round sphere  $S^n$  of diameter  $< \pi$ . For a great sphere  $S^{n-1}$  of  $S^n$ , the subcone  $K_0 \subset X$  over  $S^{n-1}$  is a locally convex set of  $X$ . This shows that Proposition 14.5 does not hold for a general locally convex set  $S$ . The following example also shows that Proposition 14.5 does not hold for a topological singular point  $p$ .

**Example 14.8.** Let  $\gamma$  be the isometric involution on  $S^2(1) \times \mathbb{R}^2$  defined by  $\gamma(x, t) = (R_\pi(x), -t)$ , where  $R_\pi$  denote a rotation by angle  $\pi$ . Then  $X^4 = S^2(1) \times \mathbb{R}^2 / \gamma$  is a complete Alexandrov space with nonnegative curvature with soul  $S$  isometric to the spherical suspension over  $S_\pi^1$ . Let  $p \in S$  be one of the two topological singular points of  $X^4$ . Then  $\Sigma_p(X^4)$  is isometric to the projective space of constant curvature 1, and the conclusion of Proposition 14.5 does not hold in this case.

For the proof of Proposition 14.5, we may assume  $\dim C(i) = 2$ . Let  $\Sigma_0 := \Sigma_p(\text{int } C(i))$  and

$$\Sigma_1 := \{\xi_1 \in \Sigma_p(X) \mid \angle(\xi_1, \xi_0) = \pi/2 \text{ for any } \xi_0 \in \Sigma_0\}.$$

From the construction of the soul  $S$  together with Proposition 2.1, every  $\xi \in \Sigma_p(X)$  lies on a minimal geodesic from a point of  $\Sigma_0$  to a point of  $\Sigma_1$ . Thus every element of  $K_p(X^4)$  lies on an infinite flat rectangle isometric to  $[0, \infty) \times [0, \infty)$  spanned by some pair of elements of  $\Sigma_0$  and  $\Sigma_1$ .

*Proof of Proposition 14.5.* We have to show that  $K_p(X^4)$  is isometric to the product  $K(\Sigma_0) \times K(\Sigma_1)$ , in other words,  $\Sigma_p(X)$  is isometric to the spherical join  $\Sigma_0 * \Sigma_1$ . In view of the previous observation, it suffices to show that there exists a unique minimal geodesic from each point  $\xi_0 \in \Sigma_0$  to each point  $\xi_1 \in \Sigma_1$ . Suppose that there are two minimal geodesics joining  $\xi_0$  and  $\xi_1$  with directions, say  $v_0 \in \Sigma_{\xi_0}(\Sigma_p(X))$  and  $w_0 \in \Sigma_{\xi_0}(\Sigma_p(X))$ .

Let  $c(t)$ ,  $0 \leq t \leq \ell = L(\Sigma_p(\text{int } C(i)))$ , be the arc-length parameter of  $\Sigma_p(\text{int } C(i))$  with  $c(0) = \xi_0$ . The direction  $v_0$  define a parallel field  $v_t$  along  $c(t)$  such that  $v_0, v_t$  and  $c|_{[0,t]}$  span a totally geodesic triangle surface of constant curvature 1 with vertices  $\xi_0, c(t)$  and  $\xi_1$ . Similarly  $w_0$  define a parallel field  $w_t$  along  $c(t)$ .

First suppose that  $v_\ell = v_0$  and  $w_\ell = w_0$ . Then it turns out that the surface  $\Sigma_{\xi_1}(\Sigma_p(X)) \simeq S^2$  with curvature  $\geq 1$  contains two disjoint closed geodesics of length  $\ell$ , which is impossible.

Thus we may assume that  $v_\ell \neq v_0$ . Note that  $K_{\xi_1}(\Sigma_p(X))$  is isometric to a product  $\mathbb{R} \times C_1$ , where  $C_1$  is a 2-dimensional Euclidean cone over a circle, say  $S_1^1$ , and the  $\mathbb{R}$ -factor is given by  $\Sigma_1$ . Let  $\bar{v}_0$  and  $\bar{v}_\ell$  be directions at  $\xi_1$  given by the minimal geodesics to  $\xi_0$  with directions  $v_0$  and  $v_\ell$  at  $\xi_0$  respectively. Let  $\bar{v}_t \in S_1^1$ ,  $0 \leq t \leq \ell$ , be the parallel field joining  $\bar{v}_0$  and  $\bar{v}_\ell$  determined by the geodesics joining  $\xi_0$  to  $\xi_1$  with directions  $v_t$ . Let  $\bar{v}_t$ ,  $\ell \leq t \leq \ell_0$ , be a unit speed geodesic joining  $\bar{v}_\ell$  to  $\bar{v}_0$  in  $S_1^1$ , such that  $\{\bar{v}_t\}_{0 \leq t \leq \ell_0}$  forms the closed geodesic  $S_1^1$ . Let  $\bar{c}(t) \in \Sigma_p(X)$  denote the point on the geodesic with direction  $\bar{v}_t$  such that  $d(\xi_1, \bar{c}(t)) = \pi/2$ . It is easy to see that  $\bar{c}(t) = c(t) \pmod{\ell\mathbb{Z}}$  and that  $v_{\ell_0} = v_0$ . In particular,  $\ell_0 = n\ell$  for some integer  $n \geq 2$ . Thus we have a singular surface  $F$  in  $\Sigma_p(X)$  spanned by the geodesics from  $c(t)$  to  $\xi_1$  with directions  $v_t$ ,  $0 \leq t \leq n\ell$ . Let  $D$  be a small disk in  $F - \Sigma_0$  around  $\xi_1$ . Note that  $\Sigma_0$  is a deformation retract of  $F - \text{int } D$ . Let  $\alpha \in \pi_1(F - \text{int } D) \simeq \mathbb{Z}$  be the homotopy class defined by  $\partial D$ , which is the  $n$ -th power of a generator of  $\pi_1(F - \text{int } D)$ . Let  $c_1$  be a path intersecting  $F$  with a unique point in  $\text{int } D$ . From the construction of  $F$ , it is possible to join the endpoints of  $c_1$  by a path  $c_2$  in  $\Sigma_p(X) - F$ . Let  $K$  denote the knot defined as the composition  $c_1$  and  $c_2$ . We set  $\Gamma := \pi_1(\Sigma_p(X) - K)$  and consider the homomorphisms

$$\pi_1(F - \text{int } D) \xrightarrow{i_*} \Gamma \xrightarrow{\rho} H_1(\Sigma_p(X) - K) = \Gamma/[\Gamma, \Gamma] \simeq \mathbb{Z},$$

where  $\rho$  is the natural homomorphism. The above discussion implies that  $\rho \circ i_*(\alpha) \equiv 0 \pmod{n\mathbb{Z}}$ . However since  $i_*(\alpha)$  is one of the generators

of a Wirtinger presentation of  $\Gamma$ ,  $\rho \circ i_*(\alpha)$  is a generator of  $\mathbb{Z}$  (cf.[14]), a contradiction. This completes the proof of the proposition.  $\square$

## 15. SOUL THEOREM IN DIMENSION FOUR I

Throughout this section, let  $X$  be a complete open Alexandrov space with nonnegative curvature.

**Conjecture 15.1.** Suppose in addition that  $X$  is topologically nice. Then there exists a positive number  $\epsilon$  such that

- (1)  $X$  is homeomorphic to  $\text{int } B(S, \epsilon)$ ;
- (2)  $B(S, \epsilon)$  is homeomorphic to a disk-bundle over  $S$ , called the normal bundle of  $S$ .

Conjecture 15.1 is certainly true for  $n = 3$  ([43]). Theorem 2.6 asserts that the conjecture is also true for  $n = 4$ . The purpose of Sections 15 and 16 is to prove Theorem 2.6.

The assumption on the topological niceness of  $X$  is essential in Conjecture 15.1.

**Example 15.2.** (1) Let  $S(\Sigma)$  and  $S_+(\Sigma)$  denote the spherical suspension and the half of the spherical suspension of  $\Sigma$  respectively, and let  $\Sigma^3$  denote the Poincare homology 3-sphere. Then the gluing  $X^5$  of  $S_+(S(\Sigma^3))$  and  $S(\Sigma^3) \times [0, \infty)$  along their boundaries is a complete open Alexandrov space with nonnegative curvature. Note that  $X^5$  is homeomorphic to  $\mathbb{R}^5$  and the soul of it is a point. However for any  $\epsilon > 0$ ,  $B(S, \epsilon)$  is not homeomorphic to  $D^5$  but to the closed unit cone  $K_1(S(\Sigma^3))$  over  $S(\Sigma^3)$ .

(2) Let  $X^5$  be as in (1) above. Then  $S^1 \times X^5$  is a complete open Alexandrov space with nonnegative curvature whose soul is a circle. Note that  $S^1 \times X^5$  is topologically regular. However for any  $\epsilon > 0$ ,  $B(S, \epsilon)$  is not homeomorphic to  $S^1 \times D^5$  but to  $S^1 \times K_1(S(\Sigma^3))$ .

Example 15.2 (1) (resp. (2)) shows that the topological niceness assumption in Conjecture 15.1 cannot be weakened by the assumption that  $X$  being a topological manifold (resp. being topologically regular) at least in dimension  $\geq 5$  (resp.  $\geq 6$ ). Of course, Conjecture 15.1 is metric by nature. The topological version of Conjecture 15.1 is

**Conjecture 15.3.** Let  $X$  be a complete open nonnegatively curved Alexandrov space, and suppose that  $X$  is a topological manifold. Then it is homeomorphic to a small neighborhood of a soul  $S$  of it which has the structure of open disk-bundle over  $S$ .

Example 15.2 tells us that the small neighborhood of  $S$  required in Conjecture 15.3 never be a metric one in general.

For simplicity we denote by  $SC_{n,k}$  Conjecture 15.1 for  $n = \dim X$  and  $k = \dim S$ . From Proposition 14.1, Conjecture 15.1 is true in the case of  $\dim S = n - 1$ : If  $\dim S = n - 1$ , then  $X$  is isometric to the normal bundle  $N(S)$  with the canonical metric.

For  $\dim S = 1$ , we have

**Lemma 15.4.** *If the conjecture  $SC_{n-1,0}$  is true, then  $SC_{n,1}$  is also true.*

*Proof.* Let  $X^n$  be an  $n$ -dimensional complete open Alexandrov space of nonnegative curvature which is topologically nice. Applying the splitting theorem to the universal covering space of  $X^n$ , we see that  $X^n$  is isometric to a quotient  $(\mathbb{R} \times N)/\mathbb{Z}$ , where  $N$  is an  $(n-1)$ -dimensional complete open Alexandrov space with nonnegative curvature whose soul is a point. Since  $X^n$  is topologically nice, so is  $N$ . Therefore the assumption yields  $N \simeq \mathbb{R}^{n-1}$ .  $\square$

Thus for  $n = 4$ , the remaining cases are those of  $\dim S = 0$  or  $2$ .

From now on let  $X$  be as in Theorem 2.6 unless otherwise stated. Let  $C := C(0)$  for simplicity. In this section, we consider the case of  $\dim C = 2$ .

We begin with the case of  $\dim S = 2$ , for which the essential case in the proof is that  $S \simeq S^2$  or  $S \simeq P^2$ : If  $S$  is homeomorphic to either a torus or a Klein bottle, then it is flat. Thus the universal covering space  $\tilde{X}$  splits as  $\mathbb{R} \times N$ , where  $N \simeq \mathbb{R}^2$ , and Theorem 2.6 certainly holds.

**Theorem 15.5.** *Theorem 2.6 holds in the case of  $\dim S = 2$  and  $\dim C = 2$ .*

*Proof.* Consider  $R_{\delta,r}(S) := S - B(S_\delta(S), r)$  for sufficiently small  $\delta > 0$  and  $r > 0$  so that one can apply Fibration Theorem 1.2 to  $R_{\delta,r/2}$ . Now we consider the convergence,  $B(S, \epsilon) \rightarrow S$  as  $\epsilon \rightarrow 0$ . Letting  $\varphi = \iota : S \rightarrow B(S, \epsilon)$  be the inclusion, and  $\psi : B(S, \epsilon) \rightarrow S$  be a measurable map such that  $d(\psi\varphi x, x) < 2\epsilon$  for every  $x \in B(S, \epsilon)$ , we define  $f_S : S \rightarrow L^2(S)$ ,  $f_{B(S, \epsilon)} : B(S, \epsilon) \rightarrow L^2(S)$  and  $f = f_S^{-1} \circ \pi \circ f_{B(S, \epsilon)} : B(S, \epsilon) \rightarrow S$  by the same formulae as in [47].  $f$  is  $(1 - \tau(\epsilon_2, \delta))$ -open and is an almost Lipschitz submersion([47]). Let  $0 < \epsilon_1 \ll \epsilon_2 \ll r$ , and denote by  $f_{\epsilon_1, \epsilon_2}$  the restriction of  $f$  to  $\text{int } A(S; \epsilon_1, \epsilon_2) - B(S_\delta(S), r)$ :

$$f_{\epsilon_1, \epsilon_2} : \text{int } A(S; \epsilon_1, \epsilon_2) - B(S_\delta(S), r) \rightarrow R_{\delta, r/2}(S).$$

Since each point of  $B(S, \epsilon_2) - B(S_\delta(S), r) - S$  is almost regular, it follows from Fibration Theorem 1.2 (see the final step of the proof in Section 21) that  $f_{\epsilon_1, \epsilon_2}$  is a topological submersion. For any  $x \in A(S; \epsilon_1, \epsilon_2) - B(S_\delta(S), r)$ , let  $(a_i, b_i)_{1 \leq i \leq 4}$  be a  $(4, \delta)$ -strainer of  $X$  at  $x$  such that

- (1)  $(a_i, b_i)_{i=1,2}$  gives a  $(2, \delta)$ -strainer of  $S$  at  $f(x)$ ;
- (2)  $d(S, a_3) = d(S, b_3) = d(S, x)$ .

Then  $(a_1)'_x$  and  $(a_2)'_x$  are almost orthogonal to the fibre  $F_x := f^{-1}(f(x))$ , and  $(a_3)'_x$  and  $(a_4)'_x$  are almost tangent to  $F_x$  (see [47]). Note also that  $(a_i)'_x$ ,  $1 \leq i \leq 3$ , are almost tangent to  $\{d_S = d_S(x)\}$ . This shows that  $(f, d_S, d_{a_3})$  is a  $\tau(\epsilon_2, \delta_2)$ -open and gives a bi-Lipschitz coordinates around  $x$ . Therefore  $(f, d_S) : A(S; \epsilon_1, \epsilon_2) - B(S_\delta(S), r) \rightarrow R_{\delta, r/2}(S) \times \mathbb{R}$  is a topological submersion. Thus for  $\epsilon_1 < \epsilon' < \epsilon < \epsilon_2$ ,  $(f, d_S)$  provides an  $S^1$ -bundle on  $A_{\epsilon', \epsilon, r} := f^{-1}(R_{\delta, r}(S)) \cap \{\epsilon' \leq d_S \leq \epsilon\}$ . It is clear that the fibre of  $f_{\epsilon_1, \epsilon_2}$  is homeomorphic to  $S^1 \times I$ . Letting  $\epsilon' \rightarrow 0$ , it is now obvious that  $f_\epsilon$ , the restriction of  $f$  to  $f^{-1}(R_{\delta, r}(S)) \cap \{d_S \leq \epsilon\}$ , is a locally trivial  $D^2$ -bundle whose restriction to  $R_{\delta, r}(S)$  is the identity.

**Lemma 15.6.** *For any  $p \in S_\delta(S)$ , there are  $\epsilon_p > 0$  and  $r_p > 0$  such that for every  $r \leq r_p$  and  $\epsilon$  with  $\epsilon/r \leq \epsilon_p$ ,*

- (1)  $B(p, r) \cap B(S, \epsilon)$  is a  $D^2$ -bundle over  $B(p, r; S)$ ;
- (2)  $\partial B(p, t) \cap B(S, \epsilon)$  is the union of the fibres over  $\partial B(p, t; S)$  for every  $r/2 \leq t \leq r$ .

*Proof.* Consider the convergence  $(\frac{1}{r}X, p) \rightarrow (K_p, o_p)$ ,  $r \rightarrow 0$ . By Proposition 14.5,  $K_p$  is isometric to the product of  $K_p(S)$  and a flat cone  $N_p$ . Since  $B(o_p, 1) \cap B(K_p(S), \epsilon)$  is a  $D^2$ -bundle over  $B(o_p, 1; K_p(S))$ , the lemma follows from the regularity of  $(d_{o_p}, d_{K_p(S)})$  on  $A(o_p; 1/2, 1) \cap (B(K_p(S), \epsilon) - K_p(S))$ .  $\square$

We are now going to patch the  $D^2$ -bundle structures on  $A_{\epsilon, r} := f^{-1}(R_{\delta, r}(S)) \cap \{d_S \leq \epsilon\}$  and on  $B(p, r) \cap B(S, \epsilon)$  for each  $p \in S_\delta(S)$ .

Let  $U_{\epsilon, r}(p)$  denote the component of  $B(S, \epsilon) - A_{\epsilon, 2r}$  containing  $p$ , and  $L_{\epsilon, r}(p) = f_\epsilon^{-1}(\partial B(p, 2r))$ .

**Lemma 15.7.** *For small enough  $0 < \epsilon \ll r$ ,  $U_{\epsilon, r}(p) - \text{int } B(p, r/2)$  is homeomorphic to  $L_{\epsilon, r}(p) \times I$ .*

*Proof.* Suppose the lemma does not hold, and put  $\epsilon := \epsilon_0 r$ , with  $\epsilon_0 = 10^{-10}$ . Under the convergence  $(\frac{1}{r}X, p) \rightarrow (K_p, o_p)$  as  $r \rightarrow 0$ ,  $f_\epsilon : B(S, \epsilon) - B(S_\delta(S), r) \rightarrow R_{\delta, r/2}(S)$  converges to a Lipschitz map

$$f_\infty : B(K_p(S), \epsilon_0) - B(o_p, 1) \rightarrow K_p(S) - B(o_0, 1/2).$$

It is easy to see that  $f_\infty$  is the restriction of the projection  $K_p = K_p(S) \times N_p \rightarrow K_p(S)$ , where  $N_p$  denotes a flat cone. Let  $U_{\epsilon_0, 1}(o_p)$  denote the limit of  $U_{\epsilon, r}(p)$  under the above convergence. The obviously  $U_{\epsilon_0, 1}(o_p) - \text{int } B(o_p, 1/2)$  is homeomorphic to  $L_{\epsilon_0, 1}(o_p) \times I$ , where  $L_{\epsilon_0, 1}(o_p) = f_\infty^{-1}(\partial B(o_p, 2))$ . The lemma follows from Stability Theorem 1.5.  $\square$

From the  $D^2$ -bundle structures on  $B(p, r/2) \cap B(S, \epsilon)$  and on  $L_{\epsilon, r}(p)$ , we have homeomorphisms  $\partial B(p, r/2) \cap B(S, \epsilon) \rightarrow S^1 \times D^2$  and  $L_{\epsilon, r}(p) \rightarrow S^1 \times D^2$ . In view of Lemma 15.7, these provide a gluing homeomorphism  $h : S^1 \times D^2 \rightarrow S^1 \times D^2$ . Letting  $r \rightarrow 0$  with  $\frac{\epsilon}{r} \ll 1$ , we conclude that  $h|_{S^1 \times S^1}$  is homotopic, and hence isotopic, to the identity. Therefore

we can patch the  $D^2$ -bundle structures on  $B(p, r/2) \cap B(S, \epsilon)$  and on  $\partial U_{\epsilon, r}(p)$  to get a  $D^2$ -bundle structure on  $B(S, \epsilon)$ . This completes the proof of Theorem 15.5.  $\square$

Next we consider the case of  $S \neq C$ . Recall that the critical point set of  $f_\epsilon$  is contained in  $\partial C$ , where

$$f_\epsilon(x) = d(C_\epsilon, x).$$

Therefore by [35], there is an  $f_\epsilon$ -gradient flow  $\psi$  on  $X - \partial C - C_\epsilon$ .

**Definition 15.8.** Let  $\dim X = n$ ,  $\dim C = k$ , and  $A \subset \partial C$ . We say that a subset  $V$  with  $A \subset V$  is an  $f_\epsilon$ -pseudo-gradient normal bundle over  $A$  with respect to  $\psi$  if the following conditions are satisfied:

- (1)  $V$  is a  $D^{n-k+1}$ -bundle over  $A$  with projection, say  $\pi : V \rightarrow A$ ;
- (2) Let  $\partial V \subset V$  denote the total space of the  $S^{n-k}$ -bundle induced from the  $D^{n-k+1}$ -bundle of (1). Then  $\partial V$  is a gluing of two  $D^{n-k}$ -bundles, say  $J_i, i = 0, 1$ , and an  $I \times S^{n-k}$ -bundle, say  $K$ , over  $A$  with projections  $\pi|_{J_i}$  and  $\pi|_K$  such that
  - (a) there is a neighborhood  $W$  of every  $p \in A$  in  $A$  and a locally trivializing homeomorphism  $h : \pi^{-1}(W) \rightarrow W \times I^{n-k+1}$  with  $\pi = p_1 \circ h$ , where  $p_1 : W \times I^{n-k+1} \rightarrow W$  is the projection, inducing following homeomorphisms for  $i = 0, 1$ 

$$(\pi|_{J_i})^{-1}(W) \simeq W \times (i \times I^{n-k}),$$

$$(\pi|_K)^{-1}(W) \simeq W \times (I \times \partial I^{n-k}).$$
  - (b)  $J_1 \subset \{f_\epsilon = \epsilon/2\}$ ,  $J_0 \subset \{f_\epsilon = c\epsilon\}$  for some constant  $c > 1$  and every fibre of  $\pi|_K$  gives a flow curve of  $\psi$ ;
  - (c) the flow curves of  $\psi$  are transversal to  $J_i$

The  $f_\epsilon$ -pseudo-gradient normal bundle  $V$  has *height*  $\nu > 0$  if  $d_C$  restricted to  $J_1$  takes the maximal value  $\nu$  at every point of  $\partial J_1$ . We call the subbundles  $J_i$  and  $K$ ,  $J_i$ -part and  $K$ -part of  $V$  respectively.

Note that if we are given an  $f_\epsilon$ -pseudo-gradient normal bundle over  $\partial C$  with respect to  $\psi$ , then parturbing and extending  $\psi$  along the fibres of the normal bundle, we can define a new  $f_\epsilon$ -pseudo-gradient flow defined on  $X - C_\epsilon$  (see Section 10 of [43] for the definition of pseudo-gradient flows).

**Theorem 15.9.** *Let  $\dim X = 4$ . If  $\dim C = 2$  and if  $\dim S = 0$ , then Theorem 2.6 holds.*

For the proof of Theorem 15.9, we apply a method similar to one used in [43], which actually gives a simplification of the argument there.

**Proposition 15.10.** *Under the same hypothesis as Theorem 15.9, for some  $\mu > 0$  and any small enough  $\epsilon > 0$ , there exists an  $f_\epsilon$ -pseudo-gradient normal bundle over  $\partial C$  of height  $\mu\epsilon$  with respect to some  $f_\epsilon$ -gradient flow.*

*Proof.* Applying Lemma 14.3 (3), we can take finitely many consecutive points  $p_1, \dots, p_N$  of  $\partial C$  with  $d(p_j, p_{j+1})$  small enough such that there is an  $f_\epsilon$ -pseudo-gradient normal  $D^3$ -bundle  $E$  over  $\{p_1, \dots, p_N\}$  of height  $\mu\epsilon$  with respect to some  $f_\epsilon$ -gradient flow. Let  $J_1(\widehat{p_j p_{j+1}})$  denote the  $D^2$ -bundle over the arc  $\widehat{p_j p_{j+1}} \subset \partial C$  extending  $J_1(p_j) \cup J_1(p_{j+1})$  with  $J_1(\widehat{p_j p_{j+1}}) \subset f_\epsilon^{-1}(\epsilon/2)$  and of ‘height  $\mu\epsilon$ ’. Let  $A_j$  denote the union of the  $f_\epsilon$ -flow curves contained in  $\{\epsilon/2 \leq f_\epsilon \leq c\epsilon\}$  and through the total space of the  $S^1$ -bundle induced from  $J_1(\widehat{p_j p_{j+1}})$ . Let  $J_0(p_j)$  denote the  $J_0$ -part of  $E|_{p_j}$ . Since  $(A_j \cap \{f_\epsilon = c\epsilon\}) \cup J_0(p_j) \cup J_0(p_{j+1})$  is homeomorphic to  $S^2$  and is locally flat, it bounds a domain  $B_j$  in  $\{f_\epsilon = c\epsilon\}$  homeomorphic to  $D^3$ . Now it is clear that

$$J_0(p_j) \cup J_0(p_{j+1}) \cup J_1(\widehat{p_j p_{j+1}}) \cup A_j \cup B_j$$

is homeomorphic to  $S^3$  and is locally flat, and therefore it bounds a domain homeomorphic to  $D^4$ , which defines a structure of an  $f_\epsilon$ -pseudo-gradient normal  $D^3$ -bundle over  $\widehat{p_j p_{j+1}}$ . This completes the proof.  $\square$

*Proof of Theorem 15.9.* It follows from Proposition 15.10 that

$$X \simeq \{f_\epsilon < \epsilon/2\} \simeq \{f_\epsilon < \nu\},$$

for any  $\nu \ll \epsilon$ . We have to prove  $\{f_\epsilon \leq \nu\} \simeq D^4$ . Using the  $f_\epsilon$ -flow curves, it is easy to see that  $\{f_\epsilon \leq \nu\}$  is contractible. By Freedman’s celebrated work (see[18]), it suffices to show  $\{f_\epsilon = \nu\} \simeq S^3$ . Put  $L := C \cap \{f_\epsilon = \nu\} \simeq S^1$ , and consider the distance function  $d_L$ . Since  $(f_\epsilon, d_L)$  is regular near  $\{f_\epsilon = \nu, 0 < d_L \leq \lambda\}$  with  $\lambda \ll \nu$ , it follows from a method similar to Proposition 15.10 that  $\{f_\epsilon = \nu, d_L \leq \lambda\} \simeq S^1 \times D^2$ . For simplicity, we put  $C_\mu^* := \{f_\epsilon \leq \mu\} \cap C$  with  $\mu \leq \epsilon/2$ . By a method similar to Theorem 15.5,  $\{f_\epsilon \leq \epsilon/2, d_C \leq 2\nu\}$  is a  $D^2$ -bundle over  $C_{\epsilon/2}^*$  if  $\nu \ll \epsilon$ . Let  $\pi : \{f_\epsilon < \epsilon/2, d_C \leq 2\nu\} \rightarrow C_{\epsilon/2}^*$  be the projection.

**Sublemma 15.11.**

$$\pi^{-1}\left(C_{\nu-\lambda^2/\nu}^*\right) \cap \{f_\epsilon = \nu\} \simeq D^2 \times S^1.$$

*Proof.* Note that each point of  $\{f_\epsilon \leq \epsilon/2, 0 < d_C \leq 2\nu\}$  is almost regular. For any point  $x$  of  $C_{\nu-\lambda^2/\nu}^*$  and any point  $y$  of  $\pi^{-1}\left(C_{\nu-\lambda^2/\nu}^*\right) \cap \{f_\epsilon = \nu\}$ , let  $\theta$  be the angle at  $y$  between  $(C_\epsilon)'_y$  and the fibre  $\pi^{-1}(x)$ . Since  $\pi/2 - \theta$  is bounded from below by a uniform constant depending only on  $\lambda/\nu$  and since  $\pi$  is almost Lipschitz submersion, it follows that  $\pi^{-1}(x) \cap \{f_\epsilon = \nu\}$  is a circle.  $\square$

By using the  $d_L$ -flow curves, we obtain

$$\begin{aligned} \{f_\epsilon = \nu\} - \pi^{-1}(C_{\nu-\lambda^2/\nu}^*) \\ \simeq \{f_\epsilon = \nu, d_L \leq \lambda\} \\ \simeq S^1 \times D^2. \end{aligned}$$

Thus we conclude  $\{f_\epsilon = \nu\} \simeq S^1 \times D^2 \cup D^2 \times S^1 \simeq S^3$ .  $\square$



**Theorem 15.12.** *Conjecture 15.1 is true in the case when  $\dim C = 1$  and  $\dim S = 0$ , where  $\dim X$  be arbitrary.*

The proof of Theorem 15.12 is similar to the 3-dimensional case, and hence omitted (see Section 13 of [43] for details).

## 16. SOUL THEOREM IN DIMENSION FOUR II

In this section, we shall consider the case of  $\dim C = 3 > \dim S$ .

We also assume  $\dim X = 4$ .

**Theorem 16.1.** *Let  $X$  be a 4-dimensional complete open Alexandrov space with nonnegative curvature, and suppose that it is topologically regular. If  $\dim C = 3 > \dim S$ , then  $X$  is homeomorphic to  $\mathcal{N}(\text{int } C)$ .*

**Theorem 16.2.** *Under the same hypothesis as Theorem 16.1, for some  $\mu > 0$  and any small enough  $\epsilon > 0$ , there exists an  $f_\epsilon$ -pseudo-gradient normal bundle over  $\partial C$  of height  $\mu\epsilon$  with respect to some  $f_\epsilon$ -gradient flow.*

*Proof of Theorem 16.1 assuming Theorem 16.2.* Let  $U$  be an  $f_\epsilon$ -pseudo-gradient normal bundle of  $\partial C$  with respect to some  $f_\epsilon$ -gradient flow  $\psi$ . It is possible to deform  $\psi$  in  $U$  along the fibres of  $\pi : U \rightarrow \partial C$  to obtain an  $f_\epsilon$ -pseudo-gradient flow on  $X - C_\epsilon$ . Therefore we have

$$X \simeq \{f_\epsilon < \epsilon/2\} \simeq \mathcal{N}(\text{int } C).$$

□

First we construct an  $f_\epsilon$ -pseudo-gradient normal bundle over a small neighborhood of any point  $p$  of  $\partial C$ .

Let  $c_p \geq 1$ ,  $\epsilon_{p,c}$ ,  $\delta_{p,c}$  and  $\mu_p$  be as in Lemmas 14.2 and 14.3.

**Lemma 16.3.** *For any  $p \in \partial C$  and  $c \geq c_p$ , let  $\epsilon, \delta' < \delta$  be constants as in Lemma 14.3. Then there is an  $f_\epsilon$ -pseudo-gradient normal bundle over  $\partial C \cap B(p, \delta)$  of height  $\mu_p\epsilon$ .*

*Proof.* By Lemma 14.3, we have an  $f_\epsilon$ -gradient flow  $\psi_p$  on  $A(\partial C; \epsilon/100, \epsilon/10)$  preserving  $d_p$  on  $A(p; \delta', \delta)$ . Put  $\Delta^2 := \partial C \cap B(p, \delta') \simeq D^2$ , and  $F := \partial\Delta^2 = \partial C \cap \partial B(p, \delta')$ . We first construct an  $f_\epsilon$ -pseudo-gradient normal bundle  $V$  over  $F$  with respect to  $\psi_p$  such that  $V \subset \partial B(p, \delta')$ . Since  $\dim F = 1$ , using the regularity of  $(f_\epsilon, d_p)$  we can apply the method of the proof of Theorem 15.9 (compare the gluing technique used in Section 12 of [43]) to  $F \subset \Delta^2 \subset \partial B(p, \delta')$ , and obtain the required normal bundle  $V$  over  $F$  such that

- (1)  $V \subset \partial B(p, \delta')$ ;
- (2)  $V$  has height  $\mu_p\epsilon$ .

Using the cone structure of  $B(p, \delta)$  from  $p$ , we extend  $V$  to an  $f_\epsilon$ -pseudo-gradient normal bundle  $W$  over  $\partial C \cap A(p; \delta', \delta)$  with respect to  $\psi_p$ .

Note that  $\partial J_1$  bounds two disjoint domains  $D_1$  and  $D_2$  of  $\{f_\epsilon = \epsilon/2, d_C = \mu_p \epsilon\}$  homeomorphic to  $D^2$ . Since  $\partial J_1 \cup D_1 \cup D_2$  is homeomorphic to  $S^2$  and is locally flat, it bounds a domain  $E$  of  $\{f_\epsilon = \epsilon/2, d_C \leq \mu_p \epsilon\}$  homeomorphic to  $D^3$ . Let  $G$  denote the union of all the  $\psi_p$ -flow curves in  $\{\epsilon/2 \leq f_\epsilon \leq c\epsilon\}$  meeting  $D_1 \cup D_2$ . Note  $G \simeq D^2 \times (I \times \partial I)$ .

**Sublemma 16.4.**  *$\partial(V \cup E \cup G)$  is contained in a domain of  $f_\epsilon^{-1}(c\epsilon)$  homeomorphic to  $\mathbb{R}^3$ .*

*Proof.* Under the convergence  $(\frac{1}{\delta}X, p) \rightarrow (K_p, o_p)$ , the level set  $\{f_{\epsilon=\epsilon'\delta} = c\epsilon\}$  converges to  $\{f_{\infty, \epsilon'} = c\epsilon'\}$ . Since  $\Sigma_p \simeq S^3$  is a topological manifold, the sublemma obviously follows.  $\square$

Since  $V \cup E \cup G \simeq D^3$  and  $\partial(V \cup E \cup G)$  is locally flat, it follows from Sublemma 16.4 that  $\partial(V \cup E \cup G)$  bounds a domain  $H$  in  $f_\epsilon^{-1}(c\epsilon)$  homeomorphic to  $D^3$ . Since  $V \cup E \cup G \cup H$  is homeomorphic to  $S^3$  and is locally flat, it bounds a domain  $K$  of  $X$  homeomorphic to  $D^4$ . Therefore the  $D^2$ -bundle  $W$  over  $\partial C \cap A(p; \delta', \delta)$  extends to a required  $D^2$ -bundle structure on  $K$  over  $\partial C \cap B(p; \delta)$ .  $\square$

*Proof of Theorem 16.2.* We are going to glue those local pseudo-gradient normal bundles to construct a pseudo-gradient normal bundle over  $\partial C$ .

By the compactness of  $\partial C$ , we have

$$c := \sup_{p \in \partial C} c_p > 0, \quad \mu := \sup_{p \in \partial C} \mu_p > 0.$$

By Lemmas 14.3 and 16.3, for any  $p \in \partial C$  and  $c$  as above, there are  $\delta_{p,c} > 0$  and an  $f_\epsilon$ -pseudo-gradient normal bundle  $U_p$  over  $\partial C \cap B(p, \delta_{p,c})$  of height  $\mu\epsilon$ , where  $\epsilon$  is any sufficiently small positive number. Choose  $p_1, \dots, p_N$  of  $\partial C$  such that  $\{B(p_i, \delta_i)\}_{1 \leq i \leq N}$  covers  $\partial C$ . Let  $K$  be a sufficiently fine triangulation of  $\partial C$  by Lipschitz curves with  $\{p_i\}_{i=1}^N \subset K^0$  all of whose simplices of  $K$  have diameters less than  $\min_{1 \leq i \leq N} \delta_{p_i, c}$ , where  $K^j$  denotes the  $j$ -skeleton of  $K$ ,  $0 \leq j \leq 2$ . Take  $\delta'_{p_i, c}$  small enough compared with  $\min_{1 \leq i \leq N} d(p_i, K^0 - \{p_i\})$ . With the above choice of  $p_i$  and  $\delta'_{p_i, c} < \delta_{p_i, c}$ , take  $\epsilon$  small enough compared with  $\min \delta'_{p_i, c}$ . Now  $U_{p_i}$  is an  $f_\epsilon$ -pseudo-gradient normal bundle over  $\partial C \cap B(p, \delta_{p_i, c})$  with respect to some  $f_\epsilon$ -gradient flow  $\psi_{p_i}$  on  $X - \partial C - C_\epsilon$ . By an argument similar to that of Lemma 16.3, it suffices to construct to an  $f_\epsilon$ -pseudo-gradient normal bundle over  $K^1$  of height  $\mu\epsilon$ .

Let us suppose that we have already constructed an  $f_\epsilon$ -pseudo-gradient normal bundle  $U$  over  $L \subset K^1$  of height  $\mu\epsilon$  such that  $U$  restricted to a small neighborhood of each vertex of  $L$  is defined by the restriction of the bundle  $U_{p_i}$  for some  $p_i$ . Let  $\sigma \in K^1 - L$ . We shall extend  $U$  to an  $f_\epsilon$ -pseudo-gradient normal bundle  $V$  over  $L \cup \sigma$  of height  $\mu\epsilon$ . Suppose  $\sigma \subset B(p_1, \delta_1) \cap B(p_2, \delta_2)$ . Let  $x_0$  and  $y_0$  be the endpoints of  $\sigma$ , and take  $x_1, y_1, y_2 \in \text{int } \sigma$  with  $x_0 < x_1 < y_1 < y_2 < y_0$ . Suppose  $\sigma \cap L$  is nonempty and consists of a point, say  $x_0$ . The other cases are similar and hence omitted. We may assume that  $U = U_{p_1}$  on a neighborhood

of  $x_0$  in  $L$ . Let  $V_1$  and  $V_2$  be the  $f_\epsilon$ -pseudo-gradient normal bundles over  $\sigma|_{[x_0, x_1]}$  and  $\sigma|_{[y_1, y_0]}$  of height  $\mu\epsilon$  determined by the restriction of the bundles  $U_{p_1}$  and  $U_{p_2}$  respectively.

We are going to glue  $V_1$  and  $V_2$ . Let  $W_1 \supset V_1$  and  $W_2 \supset V_2$  be  $f_\epsilon$ -pseudo-gradient normal bundles over normal (tubular) neighborhoods of  $\sigma|_{[x_0, x_1]}$  and  $\sigma|_{[y_1, y_0]}$  of height  $\mu\epsilon$  determined by  $U_{p_1}$  and  $U_{p_2}$  respectively. Take an  $f_\epsilon$ -gradient flow  $\psi_{p_1 p_2}$  on  $X - \partial C - C_\epsilon$  such that it coincides with  $\psi_{p_j}$  on  $W_j$ ,  $j = 1, 2$ . Let  $\tilde{V}_1$  be the  $f_\epsilon$ -pseudo-gradient normal bundle over  $\sigma|_{[x_0, y_1]}$  of height  $\mu\epsilon$  determined by  $U_{p_1}$  and extending  $V_1$ . Let  $\tilde{W}_1 \supset W_1$  be the  $f_\epsilon$ -pseudo-gradient normal bundle over a normal (tubular) neighborhood of  $\sigma|_{[x_0, y_1]}$  in  $\partial C$  determined by the restriction of  $U_{p_1}$ . Let  $H := \tilde{W}_1 \cap \{f_\epsilon = c\epsilon\}$ . By definition,

$$\tilde{W}_1 \simeq H \times I,$$

via a homeomorphism induced from  $(U_{p_1}, \psi_{p_1})$ , where  $I := [\epsilon/2, c\epsilon]$ . Consider the family  $\mathcal{S} := \{H \times t \mid t \in I\}$ . Let  $H' \subset H$  and  $I' \subset I$  be such that  $H' \times I'$  provides a small neighborhood of  $\partial C \cap \tilde{W}_1$ . Note that any  $f_\epsilon$ -pseudo-gradient flow  $\psi$  defines an embedding  $f_\psi : H \times I \rightarrow H_0 \times I$  preserving  $\mathcal{S}$  on  $H \times I - H' \times I'$ , where  $H_0$  is an open set of  $\{f_\epsilon = c\epsilon\}$  containing the closure of  $H$ . By using the topological Morse theory in [44], one can slightly deform  $\psi$  (on  $H' \times I'$ ) to an  $f_\epsilon$ -pseudo-gradient flow  $\psi'$  such that  $f_{\psi'}$  preserves  $\mathcal{S}$  on  $H \times I$ . Consider the  $J_0$ -subbundles in  $\{f_\epsilon = c\epsilon\}$ , denoted  $J_0(\tilde{V}_1)$  and  $J_0(V_2)$ , of  $\tilde{V}_1$  and  $V_2$  respectively. Take a rectangle  $R$  in  $\{f_\epsilon = c\epsilon\}$  by which  $J_0(\tilde{V}_1)$  and  $J_0(V_2)$  are connected in such a way that

- (1)  $R \cap J_0(\tilde{V}_1) = \pi_{\tilde{V}_1}^{-1}(y_1) \cap J_0(\tilde{V}_1)$ ;
- (2)  $R \cap J_0(V_2) = \pi_{V_2}^{-1}(y_2) \cap J_0(V_2)$ ;
- (3)  $\hat{H} := J_0(\tilde{V}_1) \cup R \cup J_0(V_2) \simeq I^2$ ,

where  $\pi_{\tilde{V}_1}$  and  $\pi_{V_2}$  denote the bundle projections of  $\tilde{V}_1$  and  $V_2$  respectively. The union  $\hat{V}$  of all the flow curves of  $\psi'_{p_1 p_2}$ , a deformation of  $\psi_{p_1 p_2}$  as above, through  $\hat{H}$  in  $\{\epsilon/2 \leq f_\epsilon \leq c\epsilon\}$  provides a gluing of  $V_1$  and  $V_2$ . Taking a smaller  $\mu > 0$  if necessary, we finally obtain the required  $f_\epsilon$ -pseudo-gradient normal bundle  $V \subset \hat{V}$  over  $L \cup \sigma$  of height  $\mu\epsilon$ . This completes the proof of Theorem 16.2.  $\square$

**Theorem 16.5.** *Theorem 2.6 holds in the case of  $\dim C = 3$ .*

*Proof.* This immediately follows from Theorem 16.1 and the following proposition.  $\square$

**Proposition 16.6.** *Let  $C$  be a 3-dimensional compact Alexandrov space with nonnegative curvature and with boundary. If  $C$  is a topological manifold, then  $C$  is homeomorphic to the normal bundle of the soul of  $C$ .*

*Proof.* Take a small  $\epsilon > 0$  with  $C_\epsilon \simeq C$  (Theorem 5.14). Then one can apply the method of [43] to obtain that  $C_\epsilon$  is homeomorphic to the normal bundle of the soul of  $C$ .  $\square$

## 17. THE CLASSIFICATION OF NONNEGATIVELY CURVED ALEXANDROV THREE-SPACES WITH BOUNDARY

In this section, we assume  $X^n$  to be an  $n$ -dimensional complete nonnegatively curved Alexandrov space with boundary, and give a classification of such a space in dimension three.

**Proposition 17.1.** *Suppose that  $X^n$  has nonempty boundary. If  $\dim S = n - 1$  and if  $X^n$  is a topological manifold, then  $X^n$  is isometric to either  $S \times [0, \infty)$  or an  $I$ -bundle over  $S$  for some closed interval  $I$ .*

*Proof.* By Proposition 14.1, for any point  $p \in S$ ,

- (1)  $\Sigma_p(X)$  is isometric to the spherical suspension over  $\Sigma_p(S)$ ;
- (2) for the directions  $\xi_\pm \in \Sigma_p(X)$  perpendicular to  $\Sigma_p(S)$  satisfying  $\angle(\xi_+, \xi_-) = \pi$ , there exist maximal geodesics  $\gamma_\pm : [0, \ell_\pm) \rightarrow \text{int } X$  with  $\dot{\gamma}_\pm(0) = \xi_\pm$ .

Note that  $\ell_\pm$  does not depend on the particular choice of  $p \in S$ , and that  $\gamma_\pm(\ell_\pm)$  (if  $\ell_\pm < \infty$ ) belongs to  $\partial X$ . Proposition 14.1 then implies that if the normal bundle  $N(S)$  is nontrivial, then  $X^n$  is isometric to a twisted product of  $S$  and  $I$  for some closed interval  $I$ , and that if  $N(S)$  is trivial, then  $X^n$  is isometric to either  $S \times I$  or  $S \times [0, \infty)$ .  $\square$

**Proposition 17.2.** *Suppose that  $X^n$  has nonempty boundary. If  $\dim S = 1$ , then  $X^n$  is isometric to a quotient  $(\mathbb{R} \times X_0^{n-1})/\Lambda$ , where  $\Lambda \simeq \mathbb{Z}$  and  $X_0^{n-1}$  is a complete, contractible Alexandrov space with nonnegative curvature and with boundary. Topologically,  $X^n$  is a  $X_0^{n-1}$ -bundle over  $S^1$ .*

*Proof.* Just apply the splitting theorem to the universal cover of  $X^n$ .  $\square$

**Theorem 17.3.** *For a complete nonnegatively curved Alexandrov space  $X^n$ , we have the following splitting:*

- (1) *If  $\partial X^n$  is disconnected, then  $X^n$  is isometric to a product  $X_0 \times I$ , where  $X_0$  is a component of  $\partial X$ ;*
- (2) *If  $\partial X^n$  is compact and connected and if  $X^n$  is noncompact, then  $X^n$  is isometric to the product  $\partial X^n \times [0, \infty)$ .*

In the Riemannian case, Theorem 17.3 was proved in [7]. However it seems to the author that the method used in [7] cannot be directly applied for the proof of Theorem 17.3(1). We make use of the notion of 1-systole, instead. For a non-simply connected space  $Y$ , let  $\text{sys}_1(Y)$  denote the 1-systole of  $Y$ , the infimum of the lengths of non-null homotopic loops in  $Y$ .

**Proposition 17.4.** *Let  $X^n$  be a noncompact, non-simply connected, complete Alexandrov space with nonnegative curvature, and let  $S$  be a soul of  $X^n$ . Then*

$$\text{sys}_1(X^n) = \text{sys}_1(S).$$

*In particular  $\text{sys}_1(X^n) > 0$ .*

*Proof.* The basic idea goes back to Sharafutdinov [41]. For any non-null homotopic loop  $\gamma$  in  $X$ , take a large compact totally convex set  $C$  such that

- (1)  $C \supset \gamma$ ;
- (2) there is a distance-decreasing retraction of  $C$  onto  $S$  (the Sharafutdinov retraction constructed in [35]),  $R : C \times [0, 1] \rightarrow C$  with  $R(\cdot, 0) = \text{identity}$ ,  $R(C, 1) = S$ .

Then obviously,  $R_1(\gamma)$  is a non-null homotopic loop in  $S$  satisfying  $L(R_1(\gamma)) \leq L(\gamma)$ .  $\square$

**Lemma 17.5.** *Let  $X^n$  be a complete nonnegatively curved Alexandrov space with disconnected boundary, and  $X_0$  and  $X_1$  be any distinct components of  $\partial X$ . Then we have  $d(X_0, X_1) > 0$ .*

*Proof.* We may assume that  $X^n$  is noncompact. Let us consider the double  $D(X)$ . Let  $c$  be any path from a point of  $X_0$  to a point of  $X_1$ , and  $D(c) \subset D(X)$  the double of  $c$ . Then  $D(c)$  is non-null homotopic in  $D(X)$  and therefore  $L(D(c)) \geq \text{sys}_1(S) > 0$  for a soul  $S$  of  $D(X)$ . This shows that  $d(X_0, X_1) \geq \text{sys}_1(S)/2$ .  $\square$

*Proof of Theorem 17.3.* The essential part is (1). Suppose that  $\partial X$  is disconnected, and let  $X_0$  and  $X_1$  be distinct components of  $\partial X$ , and consider the functions  $f_i = d(X_i, \cdot)$ , which are concave. Put  $X_i^t := f_i^{-1}([0, t])$ . Let  $t_0$  denote the supremum of those  $t$  with  $d(X_0^t, X_1^t) > 0$ . Lemma 17.5 ensures  $t_0 > 0$ . Consider the set  $C_* := f_0^{-1}([t_0, \infty)) \cap f_1^{-1}([t_0, \infty))$ . In what follows, we investigate the geometric properties of  $C_*$ . Note that  $C_*$  is a nonempty closed totally convex subset. Note also that  $C_{**} := f_0^{-1}(t_0) \cap f_1^{-1}(t_0)$  is nonempty.

We claim that for any  $x \in C_{**}$  and  $x_i \in X_i$  with  $d(x, x_i) = d(x, X_i)$ ,  $i = 1, 2$ , we have  $\angle x_0 x x_1 = \pi$ . For if  $\angle x_0 x x_1 < \pi$ , then  $d(x_0, x_1) < 2t_0$ . Letting  $y$  be the midpoint of a minimal geodesic joining  $x_0$  and  $x_1$ , we would have  $y \in X_0^{t_1} \cap X_1^{t_1}$  for some  $t_1 < t_0$ , a contradiction.

Next we show that  $\dim C_* \leq n - 1$ . Let  $x \in C_{**}$  and  $x_i \in X_i$  satisfy  $d(x, x_i) = d(x, X_i)$ ,  $i = 1, 2$ . If we set  $\Sigma^{n-2}$  to be the set of  $\xi \in \Sigma_x$  with  $\angle((x_0)'_x, \xi) = \angle((x_1)'_x, \xi) = \pi/2$ . Then the above claim implies that  $\Sigma_x$  is the spherical suspension over  $\Sigma^{n-2}$ . By the first variation formula, for any  $p \in C_*$  we have

$$(17.1) \quad p'_x \in \Sigma^{n-2},$$

and hence  $\Sigma_x(C_*) \subset \Sigma^{n-2}$  showing  $\dim C_* \leq n - 1$ .

We now show that  $C_* = C_{**}$ . For any  $x \in C_{**}$  and  $p \in C_*$ , let  $\gamma : [0, 1] \rightarrow X$  be a minimal geodesic from  $x$  to  $p$ . (17.1) yields that  $f_i(\dot{\gamma}(0)) = 0$ ,  $i = 1, 2$ . It follows from the concavity of  $f_i$  that  $f_i(\gamma(t)) \equiv t_0$  and  $p \in C_{**}$ .

Now for any  $x \in X_0$ , take a  $x' \in X_1$  satisfying  $d(x, X_1) = d(x, x')$ . Let  $y = \varphi(x)$  be a point on  $xx'$  with  $f_0(y) = f_1(y)$ . By the choice of  $t_0$ ,  $f_0(y) \geq t_0$  and  $f_1(y) \geq t_0$ . It follows that  $y \in C_* = C_{**}$ . Let  $x_0 \in X_0$  be a point of  $X_0$  with  $d(y, x_0) = d(y, X_0)$ . Then  $\angle x'yx_0 = \pi$  and therefore  $x = x_0$ . This also shows that  $y = \varphi(x)$  is uniquely determined by  $x$  and that  $\varphi : X_0 \rightarrow C_*$  is injective. Thus one can conclude that  $\dim C_* = n - 1$ .

For every two points  $x_0$  and  $x'_0$  of  $X_0$  take  $x_1, x'_1 \in X_1$  with  $d(x_0, x_1) = d(x'_0, x'_1) = 2t_0$ . By Proposition 2.1,  $x_0, x_1, x'_1, x_1$  span a totally geodesic flat rectangle, concluding that  $X$  is isometric to  $X_0 \times [0, 2t_0]$ .

Next suppose that  $\partial X$  is connected and compact and that  $X$  is noncompact. Let  $\gamma : [0, \infty) \rightarrow X$  be a ray starting from a point of  $\partial X$  and consider the Busemann function  $b_\gamma$  associated with  $\gamma$ . For any sufficiently large  $a \gg 1$ , the nonnegatively curved Alexandrov space  $W := b_\gamma^{-1}((-\infty, a])$  has disconnected boundary. It follows from the previous argument that  $W$  is isometric to  $\partial X \times [0, a]$ . Letting  $a \rightarrow \infty$  completes the proof.  $\square$

First we recall and reconstruct some 3-dimensional complete open Alexandrov spaces with nonnegative curvature which are not topological manifolds (see [43]).

**Example 17.6** ([43]). Let  $\Gamma$  be the discrete subgroup of isometries of  $\mathbb{R}^3$  generated by  $\gamma(x, y, z) = -(x, y, z)$  and  $\sigma(x, y, z) = (x + 1, y, z)$ . Then  $\mathbb{R}^3/\Gamma$  is a complete open nonnegatively curved Alexandrov space having two topological singular points.

**Example 17.7** ([43]). Let  $S$  be a nonnegatively curved Alexandrov surface homeomorphic to  $S^2$  having two essential singular points, say  $p_1$  and  $p_2$ . Cut  $S$  along a minimal geodesic joining  $p_1$  and  $p_2$ . The result of this cutting, say  $S_0$ , is a nonnegatively curved Alexandrov surface with boundary. The double  $\hat{S}$  of  $S_0$  has an obvious isometric involution  $\sigma$  such that  $\hat{S}/\sigma = S$ . Consider the  $\mathbb{Z}_2$ -action on  $\hat{S} \times \mathbb{R}$  defined by  $(x, t) \rightarrow (\sigma(x), -t)$ . The orbit space  $(\hat{S} \times \mathbb{R})/\mathbb{Z}_2$ , denoted  $L_{\text{sph}}(S)$ , has the two topological singular points  $p_1$  and  $p_2$ . This space corresponds to  $L(S_2; 2)$  in Example 9.4 of [43].

**Example 17.8** ([43]). Let  $S$  denote the double of a rectangle  $[0, a] \times [0, b]$ , and  $T$  the flat torus defined by the rectangle  $[-a, a] \times [-b, b]$ . Note that  $S = T/\sigma$  for the isometric involution  $\sigma$  on  $T$  defined by  $(x, y) \rightarrow (-x, -y)$ . Consider the  $\mathbb{Z}_2$ -action on  $T \times \mathbb{R}$  defined by  $(x, t) \rightarrow (\sigma(x), -t)$ . The orbit space  $(T \times \mathbb{R})/\mathbb{Z}_2$ , denoted  $L_{\text{tor}}(S)$ , has

four topological singular points. This space corresponds to  $L(S_4; 4)$  in Example 9.4 of [43].

**Example 17.9.** Let the flat torus  $T$  and the isometric involution  $\sigma$  on  $T$  be as in the previous example. Consider the involution  $\tau$  on  $T \times \mathbb{R}$  defined by  $(x, y) \rightarrow (-x + a, y + b)$ . Let  $\Omega \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  be the group generated by  $\sigma$  and  $\tau$ . Consider the  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ -action on  $T \times \mathbb{R}$  defined by  $(x, t) \rightarrow (\sigma(x), -t)$  and  $(x, t) \rightarrow (\tau(x), t)$ . The orbit space  $(T \times \mathbb{R})/\mathbb{Z}_2 \oplus \mathbb{Z}_2$  is doubly covered by  $L_{\text{tor}}(T/\sigma)$  and has two topological singular points. This orbit space is denoted by  $L_{\text{proj}}(S)$ , where we put  $S := T/\Omega \simeq P^2$ .

Each space of types  $L_{\text{sph}}(S)$ ,  $L_{\text{tor}}(S)$  and  $L_{\text{proj}}(S)$  is a 3-dimensional complete open Alexandrov space with nonnegative curvature which is not a topological manifold, and admits the structure of a singular line bundle; the singular fibre are the geodesic rays starting from the topological singular points. Note that the zero-section  $S$  of the singular line bundle is the unique soul in each case.

Here it should be remarked that no space of type  $L(S_1; 1)$  in Example 9.4 of [43] actually exists since some compatibility condition is not satisfied ! (see the proof of Case A-(3) of Theorem 17.13 below.) Of course, Theorem 9.6 in [43] is true for the 3-dimensional complete open Alexandrov spaces with nonnegative curvature which are topological manifolds (the generalized soul theorem), but for non-topological manifolds, it should be modified as follows:

**Theorem 17.10.** *Every 3-dimensional complete open Alexandrov space with nonnegative curvature which is not a topological manifold is either homeomorphic to one of the cone  $K(P^2)$  and  $\mathbb{R}^3/\Gamma$  in Example 17.6, or isometric to one of the spaces of types  $L_{\text{sph}}(S)$ ,  $L_{\text{tor}}(S)$  and  $L_{\text{proj}}(S)$ .*

The proof of Theorem 17.10 is identical with that of Theorem 9.6 of [43]. Compare the proof of Case A-(3) of Theorem 17.13 below.

Our concerns are 3-dimensional complete nonnegatively curved spaces with boundary. Let  $t$  be a positive number.

**Example 17.11.** (1) Define a  $\mathbb{Z}_2$ -action on  $D^2(t) \times \mathbb{R}$  by  $(x, s) \rightarrow (-x, -s)$ . Then the orbit space  $(D^2(t) \times \mathbb{R})/\mathbb{Z}_2$  is a complete noncompact nonnegatively curved Alexandrov space with boundary having a topological singular point. Note that the boundary of  $(D^2(t) \times \mathbb{R})/\mathbb{Z}_2$  is homeomorphic to a Möbius strip, and that the double of  $(D^2(t) \times \mathbb{R})/\mathbb{Z}_2$  is isometric to  $L_{\text{sph}}(S)$  with  $S = D(D^2(t)/\mathbb{Z}_2)$ ;

(2) The identification space

$$[0, t] \times \mathbb{R}^2/(t, x) \sim (t, -x)$$

is a complete noncompact nonnegatively curved Alexandrov space with boundary having a topological singular point. Note

that the boundary of this identification space is homeomorphic to  $\mathbb{R}^2$  and that the double of this identification space is isometric to  $\mathbb{R}^3/\Gamma$  in Example 17.6 up to a rescaling of metric;

- (3) For the discrete group  $\Gamma$  in Example 17.6,  $(\mathbb{R} \times D^2(t))/\Gamma$  is a compact nonnegatively curved Alexandrov space with boundary having two topological singular points.
- (4) Let  $L_{\text{sph}}^t(S)$  denote the totally convex subset of  $L_{\text{sph}}(S)$  defined as

$$L_{\text{sph}}(S) := d_S^{-1}([0, t]),$$

which is a compact nonnegatively curved Alexandrov space with boundary. Here  $d_S$  is the distance function from the soul  $S$ .  $L_{\text{tor}}^t(S)$  and  $L_{\text{proj}}^t(S)$  are defined similarly.

**Lemma 17.12.** *Let  $X^3$  be a complete noncompact Alexandrov space with nonnegative curvature and with boundary such that for a point  $p \in X^3$   $B(p, R) \simeq K_1(P^2)$  for any large  $R > 0$ . Consider one of the following conditions:*

- (1)  $B(p, R) \cap \partial X^3$  is homeomorphic to a Möbius band;
- (2)  $B(p, R) \cap \partial X^3$  is homeomorphic to  $D^2$ .

*If  $X^3$  satisfies (1) (resp. (2)), then it is homeomorphic to  $(D^2 \times \mathbb{R})/\mathbb{Z}_2$  (resp.  $[0, 1] \times \mathbb{R}^2/(1, x) \sim (1, -x)$ ).*

*Proof.* Note that the soul of  $X^3$  is a point, say  $p$ , with  $\Sigma_p \simeq P^2$ . Suppose first (1). Using the method in the proof of Assertion 9.13, we see that

$$X^3 \simeq B(p, \epsilon) - U \simeq (D^2 \times \mathbb{R})/\mathbb{Z}_2,$$

where  $U$  is an open disk of  $\partial B(p, \epsilon)$ . The proof in the case of (2) is similar and hence omitted.  $\square$

Now we are ready to state the classification result.

**Theorem 17.13.** *The 3-dimensional complete nonnegatively curved Alexandrov spaces  $X^3$  with boundaries are classified as follows:*

*If  $\partial X^3$  is disconnected, then  $X^3$  is isometric to a product  $X_0 \times I$ , where  $X_0$  is a component of  $\partial X^3$ .*

*Suppose  $\partial X^3$  is connected.*

*Case A.  $X^3$  is compact.*

- (1) *If  $\dim S = 0$ ,  $X^3$  is homeomorphic to either  $D^3$ , the unit cone  $K_1(P^2)$  or  $(\mathbb{R} \times D^2(1))/\Gamma$ ;*
- (2) *If  $\dim S = 1$ ,  $X^3$  is isometric to the form  $(\mathbb{R} \times N^2)/\Lambda$ , where  $N^2$  is homeomorphic to  $D^2$  and  $\Lambda \simeq \mathbb{Z}$ . In particular,  $X^3$  is homeomorphic to a  $D^2$ -bundle over  $S^1$ ;*
- (3) *If  $\dim S = 2$ ,  $X^3$  is isometric to one of a flat  $I$ -bundle over  $S$ ,  $L_{\text{sph}}^t(S)$ ,  $L_{\text{tor}}^t(S)$  and  $L_{\text{proj}}^t(S)$  for some  $t > 0$ .*



Case B.  $X^3$  is noncompact.

- (1) If  $\dim S = 2$ ,  $X^3$  is isometric to  $S \times [0, \infty)$ ;
- (2) If  $\dim S = 1$ ,  $X^3$  is isometric to the form  $(\mathbb{R} \times N^2)/\Lambda$ , where  $N^2$  is either homeomorphic to  $\mathbb{R}_+^2$  or isometric to  $\mathbb{R} \times I$  for a closed interval  $I$ , and  $\Lambda \simeq \mathbb{Z}$ . In particular,  $X^3$  is homeomorphic to an  $N^2$ -bundle over  $S^1$ ;
- (3) Suppose  $\dim S = 0$ . If  $X^3$  has two ends, then it is isometric to a product  $\mathbb{R} \times X_0$ , where  $X_0 \simeq D^2$ .

Suppose that  $X^3$  has exactly one end, and let  $C$  be the maximum set (possibly empty) of  $d_{\partial X^3}$ . Then  $C$  is either empty or of dimension  $\geq 1$ .

- (a) If  $C$  is empty,  $X^3$  is homeomorphic to  $\mathbb{R}_+^3$ ;
- (b) If  $\dim C = 1$ ,  $C$  is a geodesic ray and  $X^3$  is either homeomorphic to  $\mathbb{R}_+^3$  or the identification space  $[0, 1] \times \mathbb{R}^2 / (1, x) \simeq (1, -x)$ ;
- (c) If  $\dim C = 2$ ,  $C$  is homeomorphic to either  $\mathbb{R}^2$  or  $\mathbb{R}_+^2$ .
  - (i) If  $C \simeq \mathbb{R}^2$ , there exists an essential singular point  $p$  in  $C$ , and  $X^3$  is isometric to the identification space

$$[0, t] \times \hat{C} / (t, x) \simeq (t, \sigma(x)),$$

where  $t$  is the maximum of  $d_{\partial X^3}$ ,  $\hat{C}$  is the double of the result of cutting of  $C$  along a geodesic ray starting from  $p$  and  $\sigma$  is the involution of  $\hat{C}$  such that  $\hat{C}/\sigma = C$ ;

- (ii) If  $C \simeq \mathbb{R}_+^2$ ,  $X^3$  is homeomorphic to  $\mathbb{R}_+^3$ .

*Proof.* By Theorem 17.3, we assume that  $\partial X$  is connected. By the splitting theorem, we also assume  $\dim S \neq 1$ . Letting  $C$  be the maximum set of  $d_{\partial X^3}$ , we note that the topological singular points of  $X^3$  are contained in  $\text{Ext}(\text{int } X^3) \cap C$ . Let  $t$  be the maximum of  $d_{\partial X^3}$ .

Suppose  $X^3$  is compact. To discuss Case A, we use Theorem 9.6 in [43] and the argument there. First of all, the conclusion in Case A certainly holds if  $X^3$  is a topological manifold. Suppose  $X^3$  is not topological manifold. In Case A-(1), if  $\dim C = 0$ , then  $X^3$  is homeomorphic to  $K_1(P^2)$ , and if  $\dim C \geq 1$ , then  $X^3$  is homeomorphic to either  $K_1(P^2)$  or  $(\mathbb{R} \times D^2(1))/\Gamma$  depending on the number  $\leq 2$  of the topological singular points of  $\text{int } X^3$ .

In Case A-(3), first note that the number of essential singular points of  $S$  is at most 4. By Proposition 2.1,  $X^3$  is isometric to a singular flat  $I$ -bundle over  $S$  with possible singular fibres over essential singular points, say  $p$ , of  $S$ . Let  $\pi : X^3 \rightarrow S$  be the natural projection along fibres. For a small disk neighborhood  $B$  of  $p$ ,  $\pi^{-1}(B)$  is fibre-wise homeomorphic to  $(D^2 \times \mathbb{R})/\mathbb{Z}_2$  in Example 17.11. Now if  $S \simeq S^2$ ,  $X^3$  is built from some pieces of  $(D^2 \times \mathbb{R})/\mathbb{Z}_2$  by gluing along boundaries. To realize this gluing, we encounter some compatibility condition. This

observation implies that  $X$  is isometric to either  $L_{\text{sph}}^t(S)$  or  $L_{\text{tor}}^t(S)$ . If  $S \simeq P^2$ , the compatibility condition implies that the number  $m$  of the singular fibres in  $X^3$  is two. Because if  $m = 1$ , take a disk neighborhood  $B$  of the singular point  $p$  as above. Since the line bundle is regular one on  $S - B$  which is homeomorphic to a Möbius band, its restriction to  $\partial(S - B)$  must be trivial. On the other hand,  $\partial(D^2 \times \mathbb{R})/\mathbb{Z}_2$  is an open Möbius band. Therefore the compatibility condition does not hold. Now the universal cover  $\tilde{X}^3$  of  $X^3$  is isometric to  $L_{\text{tor}}^t(\tilde{S})$ , where  $\tilde{S}$  is the universal cover of  $S$ . Thus  $X^3$  must be isometric to  $L_{\text{proj}}^t(S) = L_{\text{tor}}^t(\tilde{S})/\mathbb{Z}_2$ .

Next consider Case  $B$ . If  $\dim S = 2$ , using Theorem 17.3, we easily obtain the conclusion. Suppose  $\dim S = 0$ . In view of the splitting theorem, we assume that  $X^3$  has exactly one end. If  $X^3$  is a topological manifold, then Assertion 9.13 shows  $X^3 \simeq \mathbb{R}_+^3$ . If  $C$  is empty,  $X^3$  is a topological manifold. Therefore we assume that  $C$  is nonempty. From the concavity of  $d_{\partial X^3}$ , every geodesic rays starting from a point of  $C$  is contained in  $C$ . In particular  $C$  is noncompact. Suppose  $\dim C = 1$ . Since  $X^3$  has exactly one end,  $C$  represents a geodesic ray, say  $\gamma$ . Since every point of  $\text{int } C$  is topologically regular, we assume that  $p := \partial C$  is a topological singular point. The critical point theory to the map  $(d_{\partial X^3}, d_p)$  shows that  $X^3$  is homeomorphic to  $\{d_{\partial X^3} \geq t - \epsilon_1\} \cap \{d_p < \epsilon_2\}$  for every  $0 < \epsilon_1 \ll \epsilon_2 \ll 1$ , where  $t$  denotes the maximum of  $d_{\partial X^3}$ . It is easy to see that  $B(p, \epsilon_2)$  is homeomorphic to  $\{d_{\partial X^3} \geq t - \epsilon_1\} \cap \{d_p < \epsilon_2\}$ . It follows from Stability Theorem 1.5 that the boundary of  $\{d_{\partial X^3} \leq \epsilon_1\} \cap \{d_p \leq \epsilon_2\}$  is homeomorphic to  $P^2$ . Since  $\{d_{\partial X^3} \leq \epsilon_1\} \cap \{d_p = \epsilon_2\}$  is homeomorphic to  $D^2$  for any sufficiently small  $\epsilon_1$  and  $\epsilon_2$ ,  $\partial X^3$  must be homeomorphic to an open Möbius band. From Lemma 17.12, we conclude that  $X^3$  is homeomorphic to  $(D^2 \times \mathbb{R})/\mathbb{Z}_2$ .

Suppose finally  $\dim C = 2$ . Using the Sharafutdinov retraction constructed in [35], one can prove that  $X$  is homotopic to  $C$ . If  $C$  was isometric to  $I \times \mathbb{R}$ , then  $X^3$  would have two ends contradicting the assumption. Therefore  $C$  is homeomorphic to either  $\mathbb{R}^2$  or  $\mathbb{R}_+^2$ .

Suppose  $C \simeq \mathbb{R}^2$ . By Proposition 2.1, there exist exactly two minimal geodesics from any  $x \in C - ES(C)$  to  $\partial X^3$ , and there exists a unique geodesic from any topological singular point  $x$  to  $\partial X^3$ . Thus we have the structure of a singular flat  $I$ -bundle on  $X^3$  over  $C$ , which provides a map  $\pi : \partial X^3 \rightarrow C$  whose restriction to the complement of the topological singular point set is locally isometric double covering. The number  $m$  of the topological singular points of  $X^3$  is at most two. If  $m = 2$ ,  $C$  would be isometric to the double  $D(I \times [0, \infty))$  for a closed interval  $I$ . It turns out that  $\partial X^3$  is the two points completion of a double covering space of  $C - ES(C)$ . This forces  $C - ES(C)$  to have a topological double cover by itself, which is impossible. Therefore  $m = 1$ . The conclusion follows from Proposition 2.1 with a similar argument.

Suppose  $C \simeq \mathbb{R}_+^2$ . By Proposition 11.3 of [43], any  $p \in \partial C$  is topologically regular. Thus  $X^3 \simeq \mathbb{R}_+^3$ .  $\square$

As an application of our argument, we give a classification of 3-dimensional complete Alexandrov spaces  $X^3$  with nonnegative curvature in terms of the number of extremal points. We assume that  $X^3$  has nonempty boundary. Let  $e_{\text{int } X^3}$  denote the number of extremal points in  $\text{int } X^3$ . If  $e_{\text{int } X^3} = 0$ , then  $X^3$  is a topological manifold, and the classification of such spaces is given by Theorem 17.13. Hence we assume  $e_{\text{int } X^3} > 0$ .

**Corollary 17.14.** *Let  $X^3$  be a 3-dimensional complete Alexandrov space with nonnegative curvature and with nonempty boundary. Then  $0 \leq e_{\text{int } X^3} \leq 4$  and the structure of  $X^3$  is described in terms of  $e_{\text{int } X^3}$  as follows. In the below,  $S$  is a soul of  $X^3$  and  $t$  denote the maximum of  $d_{\partial X^3}$ .*

- (1) *If  $e_{\text{int } X^3} = 4$ ,  $X^3$  is isometric to either  $D(I_1 \times I_2 \times \{x \geq 0\}) \cap \{x \leq t\}$  with some closed intervals  $I_1$  and  $I_2$ , or  $L_{\text{tor}}^t(S)$ ;*
- (2) *If  $e_{\text{int } X^3} = 3$ ,  $X^3$  is homeomorphic to either  $D^3$  or  $K_1(P^3)$ ;*
- (3) *If  $e_{\text{int } X^3} = 2$ ,  $X^3$  is either homeomorphic to one of  $D^3$ ,  $K_1(P^2)$ ,  $(\mathbb{R} \times D^2)/\Gamma$  and  $\mathbb{R}_+^3$ , or isometric to one of  $L_{\text{sph}}^t(S)$  and  $L_{\text{proj}}^t(S)$ ;*
- (4) *If  $e_{\text{int } X^3} = 1$ ,  $X^3$  is homeomorphic to one of  $D^3$ ,  $K_1(P^2)$ ,  $\mathbb{R}_+^3$ ,  $([0, 1] \times \mathbb{R}^2)/(1, x) \simeq (1, -x)$  and  $(D^2 \times \mathbb{R})/\mathbb{Z}_2$ .*

*Proof.* Let  $C$  denote the maximum set of  $d_{\partial X^3}$ . We may assume  $C$  to be nonempty because of  $e_{\text{int } X^3} > 0$ . Note also that  $\text{Ext}(\text{int } X^3) \subset \text{Ext}(C)$ . It follows that  $e_{\text{int } X^3} \leq 4$  and that if  $e_{\text{int } X^3} \geq 3$ , then  $C$  is a compact surface, and  $X^3$  is also compact.

Suppose  $e_{\text{int } X^3} = 4$ . Then  $C$  is isometric to either a rectangle  $I_1 \times I_2$  or the double  $D(I_1 \times I_2)$ . If  $C = I_1 \times I_2$ , the assumption  $e_{\text{int } X^3} = 4$  forces all the boundary points  $p$  of  $C$  to be one-normal points in the sense that there is a unique direction at  $p$  normal to  $C$ . This implies that  $X^3$  is isometric to  $D(I_1 \times I_2 \times \{x \geq 0\}) \cap \{x \leq t\}$ . If  $C = D(I_1 \times I_2)$ , it is the soul  $S$ , and  $X^3$  is a singular flat  $I$ -bundle over  $S$  with four singular fibres. Thus  $X^3$  must be isometric to  $L_{\text{tor}}^t(S)$ .

Suppose  $e_{\text{int } X^3} = 3$ . If  $C$  had no boundary, then  $X^3$  would be a singular flat  $I$ -bundle over  $S$  with three singular fibres, which is impossible. Therefore  $C$  has nonempty boundary. In view of Lemma 9.11, if there is no topological singular point, then  $X^3 \simeq D^3$  because the soul of  $X^3$  is a point. If there is a topological singular point,  $C$  must be the result of cutting of the double  $D(I^2)$  of a square  $I^2$  along the diagonals, and the interior singular point  $p$  of  $C$  is the unique topological singular point. Thus Theorem 17.13 yields  $X^3 \simeq K_1(P^2)$ .

The cases  $e_{\text{int } X^3} \leq 2$  also follows from Theorem 17.13 together with an argument similar to the above argument, and hence we omit the detail.  $\square$

- Remark 17.15.* (1) In the case when  $X^3$  is a complete open Alexandrov space with nonnegative curvature, considering the sublevel set  $\{b \leq t\}$  with large  $t > 0$ , one can reduce the classification problem for  $X^3$  as in Theorem 17.13 to that of a compact space with boundary;
- (2) For  $n$ -dimensional compact Alexandrov spaces with nonnegative curvature, the total number of extremal points does not exceed  $2^n$  ([37]). Recently, the metric classification of nonnegatively curved spaces with maximal extremal points has been proved in [45].

### Part 3. Equivariant fibration-capping theorem

#### 18. PRELIMINARIES

Let  $X$  be a  $k$ -dimensional complete Alexandrov space with curvature  $\geq -1$ . Now suppose that  $\partial X$  is nonempty.

Let a Lie group  $G$  act on  $X$  as isometries. Note that the group of isometries of  $X$  is a Lie group ([21]). Later on we shall assume that  $X/G$  is compact.

First we discuss the convergence in  $G$ .

Let  $\Sigma(X)$  denote the union of  $\Sigma_p(X)$  when  $p$  runs over  $X$ . We topologize  $\Sigma(X)$  by the following convergence: Let  $v_n \in \Sigma_{p_n}(X)$  and  $v \in \Sigma_p(X)$ . Then  $v_n$  converges to  $v$  if and only if  $p_n \rightarrow p$  and  $\angle(v_n, \xi_n) \rightarrow \angle(v, \xi)$  for any  $x \in X - \{p, p_n\}$  and  $x_n$  with  $\gamma_{p_n, x_n} \rightarrow \gamma_{p, x}$ , where  $\xi_n = \dot{\gamma}_{p_n, x_n}(0)$  and  $\xi = \dot{\gamma}_{p, x}(0)$ . Note that  $\Sigma(X)$  is not necessary locally compact.

The following lemma shows that the action of  $G$  on  $\Sigma(X)$  is continuous.

**Lemma 18.1.** *A sequence  $g_n$  in  $G$  converges to  $g$  if and only if  $g_n p \rightarrow gp$  and  $g_n^{-1} v_n \rightarrow g^{-1} v$  for any  $p \in X$ ,  $v_n \in \Sigma_{g_n p}$  and  $v \in \Sigma_{gp}$  with  $v_n \rightarrow v$ .*

*Proof.* We may assume  $g$  to be the identity. Let  $g_n \rightarrow 1$ , and suppose that  $\gamma_{g_n p, x_n} \rightarrow \gamma_{p, x}$  and  $v_n \in \Sigma_{g_n p}$  converges to  $v \in \Sigma_p$ . Then from definition,  $\angle(g_n^{-1} v_n, (x_n)_p') = \angle(v_n, (g_n x_n)_{g_n p_n}') \rightarrow \angle(v, x_p')$ . The converse is obvious.  $\square$

For any  $p$ ,  $G_p$  denotes the isotropy subgroup of  $G$  at  $p$ .

**Lemma 18.2.** *For each  $p \in X$ , there is a nonnegative integer  $\ell(p)$  together with the  $G_p$ -invariant isometric splitting  $K_p(X) = \mathbb{R}^{\ell(p)} \times K_p'$  satisfying*

- (1)  $K_p(Gp) = \mathbb{R}^{\ell(p)}$ ;
- (2)  $\{K_p(X) = \mathbb{R}^{\ell(p)} \times K_p'\}_{p \in X}$  are the  $G$ -invariant splittings.

*Proof.* Let  $\Sigma_p(Gp)$  denote the set of all  $\xi \in \Sigma_p$  such that there exists a sequence  $g_n \in G$  satisfying  $g_n p \rightarrow p$  and  $(g_n p)_p' \rightarrow \xi$ . First we show that

for any  $\xi \in \Sigma_p(Gp)$  there exists an  $\eta \in \Sigma_p(Gp)$  such that  $\angle(\xi, \eta) = \pi$ . Let  $g_n \in G$  be as above. Taking a subsequence if necessary, we may assume  $g_n \rightarrow g$  for some  $g \in G$ . Considering  $g^{-1}g_n$ , we may also assume that  $g_n$  converges to the identity. Put  $p_n := g_np$  and take  $q_n \in X$  with  $\tilde{Z}pp_nq_n > \pi - \epsilon_n$ ,  $\lim \epsilon_n = 0$ . Passing to a subsequence, we may assume that  $(g_n^{-1}q_n)'_p$  converges to a direction  $\xi_0 \in \Sigma_p$ . In view of  $g_n \rightarrow 1$ ,  $(q_n)'_{p_n} \rightarrow \xi_0$ . We conclude  $\xi_0 = \xi$  as follows: Putting  $\xi_n := (p_n)'_p$ , we have for any  $x \in X - \{p\}$

$$\begin{aligned} |\angle(\xi, x'_p) - \angle(\xi_0, x'_p)| &\leq |\angle(\xi, x'_p) - \angle(\xi_n, x'_p)| \\ &\quad + |\angle(\xi_n, x'_p) - \tilde{Z}p_npx| + |\tilde{Z}p_npx - (\pi - \tilde{Z}pp_nx)| \\ &\quad + |\pi - \tilde{Z}pp_nx - \tilde{Z}q_np_nx| + |\tilde{Z}q_np_nx - \angle q_np_nx| \\ &\quad + |\angle q_np_nx - \angle(\xi_0, x'_p)| < \epsilon_n, \end{aligned}$$

where  $\lim \epsilon_n = 0$ . This implies  $\xi = \xi_0$ . It follows that  $(g_n^{-1}p)'_p$  converges to  $-\xi$ .

By the above argument together with the splitting theorem,  $\Sigma_p(Gp)$  spans a Euclidean space of dimension, say  $\ell(p)$ , in  $K_p$ , and we have a  $G_p$ -invariant isometric splitting  $K_p(X) = \mathbb{R}^{\ell(p)} \times K'_p$ .

We show  $\mathbb{R}^{\ell(p)} = K_p(Gp)$ . Suppose the contrary. Since  $K_p(Gp)$  is closed in  $K_p$ , we can find sequences  $p_n \in Gp$  and  $q_n \in X - Gp$  such that

- (a)  $\angle((q_n)'_p, \mathbb{R}^{\ell(p)}) \rightarrow 0$ ;
- (b)  $d(q_n, p_n) = d(q_n, Gp)$ ;
- (c)  $(p_n)'_p$  and  $(q_n)'_p$  converge to elements of  $\Sigma_p(Gp)$  and  $\Sigma_p - \Sigma_p(Gp)$  respectively.

We may assume that  $p_n = g_np$  with  $g_n \rightarrow 1$ . Putting  $v_n := (q_n)'_{p_n}$ , we may assume  $g_n^{-1}v_n$  converges to a direction  $w$  in  $\Sigma_p$ . By  $g_n \rightarrow 1$ , we obtain  $v_n \rightarrow w$ .

Now we claim that  $w \in \mathbb{R}^{\ell(p)}$ . Otherwise, since  $w$  is perpendicular to any element of  $\Sigma_p(Gp)$ ,  $w$  must be perpendicular to  $\mathbb{R}^{\ell(p)}$ . Choose a point  $x$  in the direction  $w$ . Set  $s := d(p, x)$ ,  $\epsilon_n := d(p, p_n)$  and  $\delta_n := d(p_n, q_n)$ . Note that

$$\begin{aligned} d(x, p_n) &= \sqrt{s^2 + \epsilon_n^2} + o(\epsilon_n), \\ d(p_n, q_n) &= \sqrt{\epsilon_n^2 + \delta_n^2} + o(\epsilon_n), \\ d(x, q_n) &= \sqrt{s^2 + \epsilon_n^2 + \delta_n^2} + o(\epsilon_n), \end{aligned}$$

where  $\lim o(\epsilon_n)/\epsilon_n = 0$ , which implies

$$\angle(v_n, (x)'_{p_n}) \geq \tilde{Z}xp_nq_n \geq \pi/2 - O(\epsilon_n),$$

where  $\lim O(\epsilon_n) = 0$ . Since  $\angle(w, x'_p) = 0$ , this contradicts the convergence  $v_n \rightarrow w$ .

It turns out that  $K_p(Gp)$  is contained in the hyperplane of  $\mathbb{R}^{\ell(p)}$  perpendicular to  $w$ , which is also a contradiction to the definition of  $\ell(p)$ .

(2) is obvious.  $\square$

In view of  $K_p = \mathbb{R}^{\ell(p)} \times K'_p$ , we consider

$$L_p := \exp_p(K'_p \cap B(o_p, \epsilon))$$

for a sufficiently small  $\epsilon > 0$ , where  $\exp_p : K'_p \cap B(o_p, \epsilon) \rightarrow X$  is defined by using quasigeodesics (see [39]). Note that  $L_p$  is  $G_p$ -invariant. Let  $G \times_{G_p} L_p$  denote the orbit space of  $G \times L_p$  by the diagonal  $G_p$ -action defined as  $h(g, x) = (gh^{-1}, hx)$ . Remark that  $G \times_{G_p} L_p$  has a natural  $G$ -action and is an  $L_p$ -bundle over  $G/G_p \simeq Gp$ .

We assume the following:

**Assumption 18.3.** For each  $p \in X$ ,  $L_p$  gives a *slice* at  $p$ . Namely  $U_p := GL_p$  provides a  $G$ -invariant tubular neighborhood of  $Gp$  which is  $G$ -isomorphic to  $G \times_{G_p} L_p$ .

In the Riemannian case, Assumption 18.3 is satisfied (cf. [3]). The author believes that Assumption 18.3 is satisfied in the present general case, but he does not know the proof yet. Note that Assumption 18.3 is automatically satisfied if  $G$  is discrete.

Let  $M^n$  be another complete Alexandrov space with curvature  $\geq -1$ , and let  $G$  also act on  $M^n$  as isometries. Let  $d_{e.GH}((M, G), (X, G))$  denote the equivariant Gromov-Hausdorff distance between the  $G$ -spaces (cf. [20] for the definition).

The following theorem is an equivariant-version of Theorem 1.2. See Section 1 for the notations below.

**Theorem 18.4** (Equivariant Fibration-Capping Theorem). *Let  $X$  and  $G$  be as above satisfying Assumption 18.3. We also assume that  $X/G$  is compact. Given  $k$  and  $\mu > 0$  there exist positive numbers  $\delta = \delta_k$ ,  $\epsilon_{X,G}(\mu)$  and  $\nu = \nu_{X,G}(\mu)$  satisfying the following : Let  $Y \subset R_\delta^D(X)$  be a  $G$ -invariant closed domain such that  $\delta_D\text{-str.rad}(Y) > \mu$ . Let  $M$  be an  $n$ -dimensional complete Alexandrov space with curvature  $\geq -1$  and with  $M = R_{\delta_n}(M)$ . Suppose  $d_{e.GH}((M, G), (X, G)) < \epsilon$  for some  $\epsilon \leq \epsilon_{X,G}(\mu)$ . Then there exists a  $G$ -invariant closed domain  $N \subset M$  and a  $G$ -invariant decomposition*

$$N = N_{\text{int}} \cup N_{\text{cap}}$$

*of  $N$  into two closed domains glued along their boundaries and a  $G$ -equivariant Lipschitz map  $f : N \rightarrow Y_\nu$  such that*

- (1)  $N_{\text{int}}$  is the closure of  $f^{-1}(\text{int}_0 Y_\nu)$ , and  $N_{\text{cap}} = f^{-1}(\partial_0 Y_\nu)$ ;
- (2) the restrictions  $f|_{N_{\text{int}}} : N_{\text{int}} \rightarrow Y$  and  $f|_{N_{\text{cap}}} : N_{\text{cap}} \rightarrow \partial Y$  are
  - (a) locally trivial fibre bundles;
  - (b)  $\tau(\delta, \nu, \epsilon/\nu)$ -Lipschitz submersions.

In the proof of Theorem 18.4 below, we generalize the argument in [47]. Although the basic procedures are similar in several steps, we give the proof for readers' convenience.

The following result is of fundamental significance to describe the basic local properties of a neighborhood of a strained point.

**Lemma 18.5** ([6]). *There exists a positive number  $\delta_k$  satisfying the following: Let  $(a_i, b_i)$  be an  $(k, \delta)$ -strainer in  $D(X)$  at  $p \in X$  with length  $\geq \mu_0$  and with  $\delta \leq \delta_k$ . Then the map  $f : X \rightarrow \mathbb{R}^k$  defined by*

$$f(x) = (d(a_1, x), \dots, d(a_k, x))$$

*provides a  $\tau(\delta, \sigma)$ -almost isometry of the metric ball  $B(p, \sigma; X)$  onto an open subset of  $\mathbb{R}^k$  or  $\mathbb{R}_+^k$ , where  $\sigma \ll \mu_0$ .*

From now on we assume

$$(18.1) \quad \delta_{D\text{-str.rad}}(X) > \mu_0$$

for a fixed  $\mu_0 > 0$  and a small  $\delta > 0$ .

The purpose of the rest of this paper is devoted to prove Theorem 18.4 in the case of

$$Y = R_\delta^D(X) = X.$$

The general case is similar, and hence the proof will be omitted.

By definition, we may assume that for every  $p \in X$  there exists an admissible  $(k, \delta)$ -strainer  $(a_i, b_i)$  of length  $> \nu_0/2$  at all points in  $B_p(\sigma)$ .

**Lemma 18.6.** *Under the situation above,*

- (1) *for every  $q, r, s \in B(p, \sigma)$  with  $1/100 \leq d(q, r)/d(q, s) \leq 1$ , we have  $|\angle rqs - \tilde{\angle} rqs| < \tau(\delta, \sigma)$ ;*
- (2) *for every  $q \in X$  with  $\sigma/10 \leq d(p, q) \leq \sigma$  and for every  $x \in X$  with  $d(p, x) \ll \sigma$ , we have*

$$|\angle xpq - \tilde{\angle} xpq| < \tau(\delta, \sigma, d(p, x))/\sigma;$$

- (3) *if  $d(p, \partial X) \geq 2\sigma$ , then for every  $q \in B_p(\sigma)$  and for every  $\xi \in \Sigma_q$ , there exist points  $r, s$  such that  $d(q, r) \geq \sigma$ ,  $d(q, s) \geq \sigma$  and*

$$\angle(\xi, r'_q) < \tau(\delta, \sigma), \quad \tilde{\angle} rqs > \pi - \tau(\delta, \sigma);$$

- (4) *if  $d(p, \partial X) \leq 2\sigma$ , then for every  $q \in B_p(\sigma)$  and for every  $\xi \in \Sigma_q$  with  $\angle(\xi, (a_k)'_q) \leq \pi/2$ , there exist points  $r \in X$  and  $s \in D(X)$  such that  $d(q, r) \geq \sigma$ ,  $d(q, s) \geq \sigma$  and*

$$\angle(\xi, r'_q) < \tau(\delta, \sigma), \quad \tilde{\angle} rqs > \pi - \tau(\delta, \sigma).$$

*Proof.* (1) follows from Lemma 18.5. (2) follows from Lemma 5.6 in [6]. For the proof of (3) and (4), see [47].  $\square$

We consider  $X_\nu := \{x \in X \mid d(x, \partial X) \geq \nu\}$ .

By contradiction argument, one can prove the following two lemmas.

**Lemma 18.7.** *There exist positive numbers  $\delta \ll 1/k$  and  $\nu \ll \sigma \ll \mu_0$  such that for every  $p \in \partial X_\nu$  and  $x \in \partial X_\nu$  with  $\sigma/10 \leq d(p, x) \leq \sigma$ , there exists  $y \in \partial X_\nu$  with  $\sigma/10 \leq d(p, y) \leq \sigma$  such that  $\angle xpy > \pi - \tau(\delta, \sigma, \nu)$ .*

**Lemma 18.8.** *There exist positive numbers  $\delta \ll 1/k$  and  $\nu \ll \sigma \ll \mu_0$  such that for every  $p \in \partial X_\nu$  and  $x \in \partial X_\nu$  with  $d(p, x) \leq \sigma$ , there exists  $y \in \partial X_\nu$  with  $\sigma/10 \leq d(p, y) \leq \sigma$  such that  $\angle(x'_p, y'_p) < \tau(\delta, \sigma, \nu)$ .*

## 19. EMBEDDING $X$ INTO $L^2(X)$

Let  $L^2(X)$  denote the Hilbert space consisting of all  $L^2$  functions on  $X$  with respect to the Hausdorff  $k$ -measure, where  $G$  acts on  $L^2(X)$  by  $g \cdot \phi(x) = \phi(g^{-1}x)$  for any  $\phi \in L^2(X)$ . In this section we study the map  $f_X : X \rightarrow L^2(X)$  defined by

$$f_X(p)(x) = h(d(p, x)),$$

where  $h : \mathbb{R} \rightarrow [0, 1]$  is a smooth non-increasing function such that

- (1)  $h = 1$  on  $(-\infty, 0]$ ,  $h = 0$  on  $[\sigma, \infty)$ ;
- (2)  $h' = 1/\sigma$  on  $[2\sigma/10, 8\sigma/10]$ ;
- (3)  $-\sigma^2 < h' < 0$  on  $(0, \sigma/10]$ ;
- (4)  $|h''| < 100/\sigma^2$ .

Remark that  $f_X$  is a  $G$ -equivariant Lipschitz map.

From now on, we use  $c_1, c_2, \dots$  to express positive constants depending only on the dimension  $k$ . First we remark that by Lemma 18.5 there exist constants  $c_1$  and  $c_2$  such that for every  $p \in X$ ,

$$(19.1) \quad c_1 < \frac{\mathcal{H}^k(B_p(\sigma))}{b_0^k(\sigma)} < c_2,$$

where  $\mathcal{H}^k$  and  $b_0^k(\sigma)$  denote the Hausdorff  $k$ -measure and the volume of a  $\sigma$ -ball in  $\mathbb{R}^k$  respectively.

We next consider the directional derivatives of  $f_X$ . For  $\xi \in \Sigma_p$ , putting

$$(19.2) \quad df_X(\xi)(x) = -h'(d(p, x)) \cos \angle(\xi, x'_p), \quad (x \in X),$$

we have

$$df_X(\xi) = \lim_{t \downarrow 0} \frac{f_X(\exp t\xi) - f_X(p)}{t} \quad \text{in } L^2(X).$$

From now on we use the following norm of  $L^2(X)$  with normalization:

$$|f|^2 = \frac{\sigma^2}{b_0^k(\sigma)} \int_X |f(x)|^2 d\mathcal{H}^k(x).$$



**Lemma 19.1.** *There exist positive numbers  $c_3$  and  $c_4$  such that*

$$c_3 < |df_X(\xi)| < c_4$$

*for every  $p \in X$  and  $\xi \in \Sigma_p$ .*

*Proof.* Use Lemma 18.6 (3), (4).  $\square$

**Lemma 19.2.** *There exist positive numbers  $c_5$  and  $c_6$  such that for every  $p, q \in X$  with  $d(p, q) \leq \sigma$ ,*

$$c_5 < \frac{|f_X(p) - f_X(q)|}{d(p, q)} < c_6.$$

*In particular  $f_X$  is injective.*

The proof is straightforward, and hence omitted.

Let  $K_p = K(\Sigma_p)$  be the tangent cone at  $p$ . We make an identification  $\Sigma_p = \Sigma_p \times \{1\} \subset K_p$ . The map  $df_X : \Sigma_p \rightarrow L^2(X)$  naturally extends to  $df_X : K_p \rightarrow L^2(X)$ . Next we show that  $df_X(K_p)$  can be approximated by a  $k$ -dimensional subspace of  $L^2(X)$ .

**Lemma 19.3.** *For any  $p \in X$ , let  $(a_i, b_i)$  be an admissible  $(k, \delta)$ -strainer at  $p$ . Taking  $\xi_i$  in  $(a_i)'_p$ , we have for any  $\xi \in \Sigma_p$ ,*

$$|df_X(\xi) - \sum_{i=1}^n c_i df_X(\xi_i)| < \tau(\delta),$$

*where  $c_i = \cos \angle(\xi_i, \xi)$ . In particular,  $df_X(\xi_1), \dots, df_X(\xi_k)$  are linearly independent in  $L^2(X)$ .*

*Proof.* Let  $\phi : \Sigma_p(D(X)) \rightarrow S^{k-1}(1)$  be the  $\tau(\delta)$ -almost isometry defined by

$$\phi(\xi) = (\cos \angle(\xi_i, \xi)) / |(\cos \angle(\xi_i, \xi))|.$$

(See [6]). It is easy to verify

$$\left| \cos \angle(\xi, \eta) - \sum_{i=1}^k c_i \cos \angle(\xi_i, \eta) \right| < \tau(\delta),$$

for every  $\eta \in \Sigma_p$ , from which the lemma follows.  $\square$

Thus if  $p \in \text{int } X$  (resp. if  $p \in \partial X$ ), then  $df_X(K_p)$  can be approximated by the  $k$ -dimensional subspace  $\Pi_p$  generated by  $\{df_X(\xi_i)\}_{1 \leq i \leq k}$  (resp. by a  $k$ -dimensional half-space of  $\Pi_p$ ). In view of Lemma 19.3, one may say that  $df_X$  is almost linear.

## 20. $G$ -INVARIANT TUBULAR NEIGHBORHOOD

In this section, we construct a  $G$ -invariant tubular neighborhood of  $f_X(X_\nu)$  in  $L^2(X)$  in a generalized sense, where  $\nu \ll \sigma$ .

Let  $G_k(L^2(X))$  be the infinite-dimensional Grassmann manifold consisting of all  $k$ -dimensional subspaces of  $L^2(X)$ .

For any  $p \in X$ , let  $K_p = \mathbb{R}^{\ell(p)} \times K'$  be as in Lemma 18.2, and let  $U_p = GL_p$  be as in Assumption 18.3.

**Lemma 20.1.** *There exists a  $G_p$ -invariant  $k$ -dimensional subspace  $\Pi_p$  of  $L^2(X)$  such that  $\angle(\Pi_p, df_X(K_p)) < \tau(\delta)$ .*

*Proof.* Let  $\Pi'$  be a  $k - \ell(p)$ -dimensional subspace of  $L^2(X)$  which is  $\tau(\delta)$ -close to  $df_X(K')$ . Then  $G_p\Pi'$  is a  $G_p$ -invariant compact subset of  $L^2(X)$ . Since  $G_p\Pi'$  can be considered as a  $G_p$ -invariant subset of  $G_{k-\ell(p)}(L^2(X))$  whose diameter is small, we can find a  $G_p$ -fixed point, say  $\Pi$ , near  $\Pi'$  by using the center of mass technique on  $G_{k-\ell(p)}(L^2(X))$ . It suffices to put  $\Pi_p := \Pi \oplus df_X(K_p(Gp))$ .  $\square$

Now fix  $p$  and take  $L_p$  so small that  $G_q \subset G_p$  for all  $q \in L_p$ . For any  $q \in L_p$  and  $x \in Gq$ , we put  $\Pi_q := \Pi_p$ ,  $\Pi_x := g(\Pi_p)$ , where  $x = gq$ . Note that  $\{\Pi_x\}_{x \in U_p}$  provides a  $G$ -invariant field of  $k$ -dimensional subspaces of  $L^2(X)$  which are  $\tau(\delta)$ -almost tangent to  $f_X(X)$ .

**Lemma 20.2.** *If  $L_p$  is sufficiently small, then  $\angle(\Pi_x, \Pi_y) \leq Cd(x, y)$  for all  $x, y \in U_p$ , where  $C = C_{U_p}$ .*

*Proof.* Since  $G$  acts on  $G_k(L^2(X))$  isometrically, the map  $\pi : G/G_p \rightarrow G_k(L^2(X))$  defined by  $\pi([g]) := g(\Pi_p)$  is Lipschitz with respect to a  $G$ -invariant metric on  $G/G_p$ . This implies that  $\angle(\Pi_p, \Pi_{gp}) \leq C_1 d([e], [g])$  for some constant  $C_1$ . Since it is straightforward to see that  $d([e], [g]) \leq C_2 d(p, gp)$  for some constant  $C_2$ , it follows that  $\angle(\Pi_p, \Pi_{gp}) \leq Cd(p, gp)$ . If  $x \in Gq$ ,  $q \in L_p$  and  $y \in Gx$ , this argument shows that  $\angle(\Pi_x, \Pi_y) \leq C'd(x, y)$  for sufficiently small  $L_p$ . Next consider the general case when  $x = gx_0$ ,  $x_0 \in g_0L_p$  and  $y \in g_0L_p$  for some  $g$  and  $g_0$  in  $G$ . We may assume that both  $x$  and  $y$  are close to  $x_0$ . Then the conclusion follows from the above argument and the fact that  $\tilde{\angle}xx_0y$  is close to  $\pi/2$ .  $\square$

**Lemma 20.3.** *For any  $p, q \in \text{int } X$  or for any  $p, q \in \partial X$ ,*

$$d_H^{L^2}(df_X(\Sigma_p), df_X(\Sigma_q)) < \tau(\delta, \sigma, d(p, q)/\sigma),$$

where  $d_H^{L^2}$  denotes the Hausdorff distance in  $L^2(X)$ .

*Proof.* First suppose that  $p, q \in \text{int } X$ . Let  $(a_i, b_i)$  be an admissible  $(n, \delta)$ -stainer in  $D(X)$  at  $p$ . For every  $\xi \in \Sigma_q$  take a point  $r \in D(X)$  satisfying  $d(q, r) \geq \sigma$  and  $\angle(\xi, r'_q) < \tau(\delta, \sigma)$ . Taking  $\xi_1$  in  $r'_p$ , we have

$$|\angle(\xi, x'_q) - \angle(\xi_1, x'_p)| < \tau(\delta, \sigma, d(p, q)/\sigma),$$

for all  $x$  with  $\sigma/10 \leq d(p, x) \leq \sigma$ . It follows that  $|df_X(\xi) - df_X(\xi_1)| < \tau(\delta, \sigma, d(p, q)/\sigma)$ .

If  $p, q \in \partial X$ , take the above  $r$  from  $X$  in place of  $D(X)$ . Then the conclusion follows from a similar argument.  $\square$

**Lemma 20.4.** *For any  $p, q \in X$  and  $\xi$  in  $q'_p$ ,*

$$(20.1) \quad \left| \frac{f_X(q) - f_X(p)}{d(q, p)} - df_X(\xi) \right| < \tau(\delta, \sigma, d(p, q)/\sigma).$$

*Proof.* Note that  $|\angle xpq - \tilde{\angle} xpq| < \tau(\delta, \sigma, d(p, q)/\sigma)$  for all  $x$  with  $\sigma/10 \leq d(p, x) \leq \sigma$  and that  $|d(x, q) - d(x, p) + t \cos \tilde{\angle} xpq| < t\tau(t/\sigma)$ ,  $t = d(p, q)$ . It follows that

$$(20.2) \quad |d(x, q) - d(x, p) + t \cos \angle(\xi, x')| < t\tau(\delta, \sigma, t/\sigma),$$

which yields (20.1).  $\square$

Let  $\sigma_1 \ll \sigma$  and let us use the simpler notation  $\tau_\delta$  to denote a positive function of type  $\tau(\delta, \sigma, \sigma_1/\sigma)$ .

Let  $\pi : X \rightarrow X/G$  be the orbit projection, and for  $\bar{p} \in \pi(\partial X)$  let  $p \in \partial X$  be a point over  $\bar{p}$ . For a  $G$ -slice  $L_p$  at  $p$  with  $\text{diam}(L_p) \leq \sigma_1$ , let  $U_p = GL_p$  be the  $G$ -invariant neighborhood of  $Gp$  as in Assumption 18.3, and  $U_{\bar{p}} := \pi(U_p)$ . We take a finite open covering  $\{U_{\bar{p}_i}\}_{i=1,2,\dots}$  of  $X/G$  as above.

**Lemma 20.5.** *Suppose that  $X/G$  is compact. Then there exists a  $G$ -equivariant Lipschitz map  $T : X \rightarrow G_k(L^2(X))$  such that*

- (1)  $\angle(T(x), \Pi_{p_i}) < \tau_\delta$  if  $x \in U_{p_i}$ ;
- (2)  $\angle(T(x), T(y)) < Cd(x, y)$ , where  $C = C_{G,X}$  is a constant depending only on the  $G$ -action on  $X$ .

*Proof.* Let  $\{\bar{\rho}_i\}$  be a partition of unity consisting of Lipschitz functions associated with  $\{U_{\bar{p}_i}\}_{i=1,2,\dots}$ , and set  $\rho_i := \bar{\rho}_i \circ \pi$ .

We use the center of mass technique on  $G_k(L^2(X))$ . For each  $x \in X$ , consider the weighted distance functions

$$\phi_x := \sum_{i \in I_x} \rho_i(x) d(\Pi_{p_i, x}, \cdot)$$

on  $G_k(L^2(X))$  with weights  $\rho_i(x)$ , where  $I_x := \{i \mid x \in U_{p_i}\}$ . Since  $\rho_i(x) = 0$  except finitely many  $i$ , all  $\Pi_{p_i, x}$  in the righthand side actually lies in some finite dimensional Euclidean space  $E$ . Since  $G_k(E)$  is totally geodesic in  $G_k(L^2(X))$ ,  $\phi_x$  has a unique minimum point, say  $T(x)$ , on  $G_k(E)$ . It is straightforward to see that  $T : X \rightarrow G_k(L^2(X))$  is  $G$ -equivariant. A convexity argument also shows that  $T$  is Lipschitz.  $\square$

Let  $G_k^*(L^2(X))$  be the Grassmann manifold consisting of all subspaces of codimension  $k$  in  $L^2(X)$ , and  $N : X \rightarrow G_k^*(L^2(X))$  the dual of  $T$ ,  $N(x) = T(x)^\perp$ , where  $T(x)^\perp$  denotes the orthogonal complement of  $T(x)$ . The angle  $\angle(N(x), N(y))$  is defined as  $\angle(N(x), N(y)) = \angle(T(x), T(y))$ .

Lemma 20.5 immediately implies

**Lemma 20.6.** *Suppose that  $X/G$  is compact. Then the map  $N : X \rightarrow G_k^*(L^2(X))$  is  $G$ -equivariant and Lipschitz with Lipschitz constant  $C = C_{G,X}$ .*

Now we consider the set  $\partial X_\nu$ . For each  $x \in \partial X_\nu$ , Let  $V_x$  denote the set of directions at  $x$  consisting of all minimal segments from  $x$  to  $\partial X$ . Since  $\text{diam}(V_x) < \tau(\delta, \sigma)$ , a center of mass technique on  $S(L^2(X)) := \{v \in L^2(X) \mid |v| = 1\}$  similar to Lemma 20.5 yields

**Lemma 20.7.** *Suppose that  $X/G$  is compact. Then there exists a  $G$ -equivariant Lipschitz map  $\mathbf{n} : \partial X_\nu \rightarrow S(L^2(X))$  such that*

- (1)  $\angle(\mathbf{n}(x), df_X(V_x)) < \tau_\delta$ ;
- (2)  $\angle(\mathbf{n}(x), \mathbf{n}(y)) < Cd(x, y)$ , where  $C = C_{G,X}$  is a constant depending only on the  $G$ -action on  $X$ .

For  $x \in \partial X_\nu$ , let us denote by  $\hat{H}(x)$  the subspace of codimension  $k-1$  generated by  $N(x)$  and  $\mathbf{n}(x)$ , and by  $H(x)$  the half space of  $\hat{H}(x)$  containing  $\mathbf{n}(x)$  and bounded by  $N(x)$ .

We consider the “normal bundle”  $\mathcal{W}$  of  $f_X(X_\nu)$  as  $\mathcal{W} := \{(x, v) \in X_\nu \times L^2(X) \mid v \in W(x)\}$ , where

$$W(x) = \begin{cases} H(x) & x \in \partial X_\nu \\ N(x) & x \in \text{int } X_\nu. \end{cases}$$

Note that  $\mathcal{W}$  is  $G$ -invariant. For  $c > 0$ , we put

$$\mathcal{W}(c) = \{(x, v) \in \mathcal{W} \mid |v| < c\}.$$

**Lemma 20.8.** *There exists a positive number  $\kappa = C$  such that  $\mathcal{W}(\kappa)$  provides a  $G$ -invariant tubular neighborhood of  $f_X(X_\nu)$ . Namely*

- (1)  $f_X(p_1) + v_1 \neq f_X(p_2) + v_2$  for every  $(p_1, v_1) \neq (p_2, v_2) \in \mathcal{W}(\kappa)$ ;
- (2) the set  $U(\kappa) = \{x + v \mid (x, v) \in \mathcal{W}(\kappa)\}$  is open in  $L^2(X)$ .

*Proof.* Suppose that  $x_1 + v_1 = x_2 + v_2$  for  $x_i = f_X(p_i)$  and  $v_i \in W(p_i)$ . We first assume  $d(p_1, p_2) \leq \sigma_1$ .

Case 1.  $p_1, p_2 \in \text{int } X_\nu$ .

Put  $K = \{x_1 + N(p_1)\} \cap \{x_2 + N(p_2)\}$ , and let  $y \in K$  and  $z \in x_2 + N(p_2)$  be such that  $|x_1 - y| = d(x_1, K)$ ,  $|x_1 - y| = |y - z|$  and that  $\angle x_1 y z = \angle(x_1 - y, N(p_2)) \leq \angle(N(p_1), N(p_2))$ . Then Lemma 20.3 implies that  $\angle x_1 y z < \tau_\delta$ . It follows from the choice of  $z$  that  $|\angle(x_1 - z, N(p_2)) - \pi/2| < \tau_\delta$ . On the other hand the fact  $\angle(x_2 - x_1, T(p_1)) < \tau_\delta$  (Lemma 20.4) also implies that  $|\angle(x_2 - x_1, N(p_2)) - \pi/2| < \tau_\delta$ . It follows that  $|x_2 - z| < \tau_\delta |x_1 - x_2|$ . Putting  $\ell = |y - x_1| = |y - z|$  and using

Lemma 20.5, we then have

$$\begin{aligned} |x_1 - z| &\leq \ell \angle x_1 y z \\ &\leq \ell \angle(T(p_1), T(p_2)) \\ &\leq \ell C |x_1 - x_2|. \end{aligned}$$

Thus we obtain  $\ell \geq (1 - \tau_\delta)/C$  as required.

Case 2.  $p_1, p_2 \in \partial X_\nu$ .

Apply the above argument to  $\hat{H}(p_i)$  instead of  $N(p_i)$ .

Case 3.  $p_1 \in \partial X_\nu$  and  $p_2 \in \text{int } X_\nu$ .

Apply the above argument to  $H(p_1)$  and  $N(p_2)$  instead of  $N(p_i)$ . Let  $K = \{x_1 + H(p_1)\} \cap \{x_2 + N(p_2)\}$ . If  $K$  meet  $x_1 + N(p_1)$ , we can apply the argument of Case 1 to  $N(p_i)$ . If  $K$  does not meet  $N(p_1)$ , it is an affine subspace parallel to  $N(p_1)$ , and let  $\hat{K}$  be the affine space generated by  $K$  and line segment from  $x_1$  to  $K$ . Then we can apply the argument of Case 1 to  $\hat{K}$  and  $N(p_2)$ . Thus we obtain (1).

Next we shall prove (2), which follows from the argument above: We assume Case 1. The other cases are similar, and hence omitted.

For any  $y \in U(\kappa)$  with  $y \in f_X(p_0) + N(p_0)$ ,  $x_0 := f_X(p_0) \in f_X(X_\nu)$  and for any  $z \in L^2(X)$  close to  $y$ , let  $T_0$  be the  $n$ -plane through  $z$  and parallel to  $T(x_0)$ , and  $y_0$  the intersection point of  $T_0$  and  $N(x_0)$ . If  $x \in f_X(X_\nu)$  is near  $x_0$ , then  $N(x)$  meets  $T_0$  at a unique point, say  $\alpha(x)$ . With the above argument, we can observe that  $\alpha$  is a homeomorphism of a neighborhood of  $x_0$  in  $f_X(\partial X_\nu)$  onto a neighborhood of  $y_0$  in  $T_0$ . Hence  $z \in U(\kappa)$  as required.

Finally we shall finish the proof of (1). Suppose that  $q_0 := f_X(p_0) + v_0 = f_X(p_1) + v_1$  for some  $p_i$  and  $v_i$  with  $d(p_0, p_1) > \sigma_1$ ,  $|v_i| < C$ . For a curve  $p_s$  joining  $p_0$  to  $p_1$ , put  $v(s, t) := (1 - t)f_X(p_s) + tq_0$ . We assert that there exists a continuous map  $V : [0, 1] \times [0, 1] \rightarrow \mathcal{W}$  such that  $\text{Exp}_{\mathcal{W}}(V(s, t)) = v(s, t)$ , yielding  $\text{Exp}_{\mathcal{W}}(V(s, 1)) = q_0$  for any  $s \in I$ , a contradiction to the previous argument. To prove this assertion, consider the set  $I \subset [0, 1]$  such that for  $t \in I$  there exists a lift  $V : [0, 1] \times [0, t] \rightarrow \mathcal{W}$  of  $v$  as above. Actually  $0 \in I$  and the previous argument shows that  $I$  is open. We define a metric of  $\mathcal{W}$  by  $d((x_1, v_1), (x_2, v_2)) := (d(x_1, x_2)^2 + |v_1 - v_2|^2)^{1/2}$ . Then the proof of Lemma 20.9 implies that

$$d((x_1, v_1), (x_2, v_2)) \leq Cd(x_1 + v_1, x_2 + v_2),$$

from which the closedness of  $I$  follows.  $\square$

Next let us study the properties of the projection  $\pi : \mathcal{W}(\kappa) \rightarrow f_X(X_\nu)$  along  $\mathcal{W}$ . By definition,  $\pi(x) = y$  if  $x \in W(y)$ .

**Lemma 20.9.** *The map  $\pi : \mathcal{W}(\kappa) \rightarrow f_X(X_\nu)$  is Lipschitz continuous. More precisely, if  $x, y \in \mathcal{W}(\kappa)$  are close to each other and  $t = |x - \pi(x)|$ , then*

- (1)  $|\pi(x) - \pi(y)|/|x - y| < 1 + \tau_\delta + Ct$ ;  
(2) if the angle between  $y - x$  and the fibre  $W(\pi(x))$  is equal to  $\pi/2$ , then

$$|(y - x) - (\pi(y) - \pi(x))| < (\tau_\delta + Ct)|x - y|.$$

*Proof.* First we prove (2).

Case 1.  $\pi(x) \in f_X(\text{int } X_\nu)$ .

Let  $N$  be the affine space of codimension  $k$  parallel to  $N_{\pi(x)}$  and through  $y$ . Let  $y_1$  and  $y_2$  be the intersections of  $N_{\pi(y)}$  and  $N$  with  $T_{\pi(x)}$  respectively. Let  $z$  be the point in  $K = N \cap N(\pi(y))$  such that  $|y_2 - z| = d(y_2, K)$ , and  $y_3 \in N(\pi(y))$  the point such that  $|y_2 - z| = |y_3 - z|$  and  $\angle y_2 z y_3 = \angle(y_2 - z, N(\pi(y))) \leq \angle(N, N(\pi(y)))$ . An argument similar to that in Lemma 20.8 yields that

$$\begin{aligned} |y_1 - y_3| &< \tau_\delta |y_1 - y_2|, \\ |y_2 - y_3|/|z - y_2| &\leq \angle(N(\pi(x)), N(\pi(y))) \leq C|\pi(x) - \pi(y)|. \end{aligned}$$

It follows that  $|y_1 - y_2| < Ct|\pi(x) - \pi(y)|$ . Furthermore the assumption implies  $|(\pi(x) - y_2) - (x - y)| < \tau_\delta |x - y|$ . Therefore we get

$$\begin{aligned} |(\pi(x) - y_1) - (x - y)| &\leq |(\pi(x) - y_1) - (\pi(x) - y_2)| + |(\pi(x) - y_2) - (x - y)| \\ &\leq |y_1 - y_2| + \tau_\delta |x - y| \\ &< Ct|\pi(x) - \pi(y)| + \tau_\delta |x - y|. \end{aligned}$$

On the other hand, since  $\angle y_1 \pi(x) \pi(y) < \tau_\delta$ ,

$$|(\pi(x) - \pi(y)) - (\pi(x) - y_1)| < \tau_\delta |\pi(x) - \pi(y)|.$$

Combining the two inequalities, we obtain that

$$|(\pi(x) - \pi(y)) - (x - y)| < (\tau + Ct)|\pi(x) - \pi(y)| + \tau_\delta |x - y|,$$

from which (2) follows.

Case 2. The  $\pi$ -image of a small neighborhood of  $x$  is contained in  $f_X(\partial X_\nu)$ .

Apply the above argument to the affine subspaces  $\hat{H}(\pi(x))$  and  $\hat{H}(\pi(y))$  in place of  $N(\pi(x))$  and  $N(\pi(y))$ .

Case 3. The other case.

This case can be reduced to the Case 1.

For (1), we assume Case 1. The other case is similar. Take  $y_0 \in N(\pi(y))$  such that  $|x - y_0| = d(x, N(\pi(y)))$ . Then (2) implies

$$\begin{aligned} \frac{|\pi(x) - \pi(y)|}{|x - y|} &\leq \frac{|\pi(x) - \pi(y)|}{|x - y_0|} \\ &\leq 1 + \tau_\delta + Ct. \end{aligned}$$

□

## 21. ALMOST LIPSCHITZ SUBMERSION

In this section, we shall prove Theorem 18.4.

Let  $M$  be as in Theorem 18.4. We assume  $d_{G-GH}((M, G), (X, G)) < \epsilon$  and  $\epsilon \ll \sigma_1, \nu$ . Let  $\varphi : X \rightarrow M$  and  $\psi : M \rightarrow X$  be  $2\epsilon$ - $G$ -approximations such that  $d(\psi\varphi(x), x) < \epsilon$ ,  $d(\varphi\psi(x), x) < \epsilon$ , where we can take such a  $\varphi$  to be measurable and  $G$ -equivariant. Then the map  $f_M : M \rightarrow L^2(X)$  defined by

$$f_M(p)(x) = h(d(p, \varphi(x))), \quad (x \in X)$$

is  $G$ -equivariant. Since  $f_M(M) \subset \mathcal{W}(\tau(\nu + \epsilon))$ , the map

$$f = f_X^{-1} \circ \pi \circ f_M : M \rightarrow X_\nu$$

is well defined, and Lemma 20.8 shows that it is  $G$ -equivariant. It also follows from Lemma 20.9 that  $f$  is a Lipschitz map.

As before,  $df_M(\xi) \in L^2(X)$ ,  $\xi \in \Sigma_p$ , is given by

$$(21.1) \quad df_M(\xi)(x) = -h'(d(p, \varphi(x))) \cos \angle(\xi, \varphi(x)'_p).$$

**Lemma 21.1.** *For every  $p, q \in M$  and  $\xi \in q'_p$ ,*

$$\left| \frac{f_M(q) - f_M(p)}{d(q, p)} - df_M(\xi) \right| < \tau(\delta, \sigma, \epsilon/\nu, d(p, q)/\sigma).$$

*Proof.* For every  $x$  with  $\sigma/10 \leq d(\psi(p), x) \leq \sigma$ , take  $y \in X$  such that  $\tilde{Z}x\psi(p)y > \pi - \tau(\delta, \sigma)$ ,  $\nu/10 \leq d(\psi(p), y) \leq \nu$ . Since  $\tilde{Z}\varphi(x)p\varphi(y) > \pi - \tau(\delta, \sigma, \epsilon/\nu)$ , it follows from an argument similar to Lemma 20.4 that

$$\begin{aligned} |d(q, \varphi(x)) - d(p, \varphi(x)) + d(q, p) \cos \angle(\xi, \varphi(x)'_p)| \\ < d(q, p) \tau(\delta, \sigma, \epsilon/\nu, d(p, q)/\sigma), \end{aligned}$$

which implies the required inequality. □

We put

$$\begin{aligned} M_{\text{int}} &:= \text{the closure of } \{p \in M \mid f(p) \in \text{int } X_\nu\}, \\ M_{\text{cap}} &:= \{p \in M \mid f(p) \in \partial X_\nu\}. \end{aligned}$$

**Lemma 21.2.** *Every  $p \in M_{\text{int}}$  (resp.  $p \in M_{\text{cap}}$ ) satisfies  $d(f(p), \psi(p)) < \tau(\epsilon)$  (resp.  $d(f(p), \psi(p)) < \tau(\nu)$ ).*

*Proof.* Every  $p \in M_{\text{int}}$  (resp.  $p \in M_{\text{cap}}$ ) satisfies  $d(f_M(p), f_X(X_\nu)) < \tau(\epsilon)$  (resp.  $d(f_M(p), f_X(X_\nu)) < \tau(\nu)$ ), from which the lemma follows immediately. □

**Lemma 21.3** (Comparison Lemma). *Let  $x_i \in M$  and  $\bar{x}_i \in X$ ,  $1 \leq i \leq 3$ , be given.*

- (1) *If  $x_1, x_2 \in M_{\text{int}}$ ,  $\nu/10 \leq d(x_1, x_2) \leq \nu$ ,  $\sigma/10 \leq d(x_2, x_3) \leq \sigma$  and  $d(f(x_i), \bar{x}_i) < \tau(\epsilon)$ ,  $1 \leq i \leq 3$ , then*

$$|\angle xyz - \angle \bar{x}\bar{y}\bar{z}| < \tau(\delta, \sigma, \epsilon/\nu);$$

- (2) *If  $x_1, x_2 \in M_{\text{cap}}$ ,  $\sigma/10 \leq d(x_1, x_2), d(x_2, x_3) \leq \sigma$ , and  $d(f(x_i), \bar{x}_i) < \tau(\epsilon)$ ,  $1 \leq i \leq 3$ , then*

$$|\angle xyz - \angle \bar{x}\bar{y}\bar{z}| < \tau(\delta, \sigma, \nu/\sigma).$$

*Proof.* We prove (1). The proof of (2) is similar and hence omitted. From the assumption, we can take a point  $\bar{w} \in X$  such that

$$(21.2) \quad \tilde{\angle} \bar{x}_1 \bar{x}_2 \bar{w} > \pi - \tau(\delta, \sigma)$$

and  $d(\bar{x}_2, \bar{w}) \geq \nu/10$ . Put  $w = \varphi(\bar{w})$ . Then

$$\angle x_1 x_2 x_3 > \angle \bar{x}_1 \bar{x}_2 \bar{x}_3 - \tau(\delta, \sigma, \epsilon/\nu),$$

$$\angle w x_2 x_3 > \angle \bar{w} \bar{x}_2 \bar{x}_3 - \tau(\delta, \sigma, \epsilon/\nu).$$

Since (21.2) implies

$$|\angle x_1 x_2 w - \pi| < \tau(\delta, \sigma, \epsilon/\nu),$$

we have the required inequality.  $\square$

We now fix  $p \in M$ , and put

$$\begin{aligned} H_p^{\text{int}} &:= \{\xi \in \Sigma_p \mid \xi \in x'_p, d(p, x) \geq \nu/10\}, \quad (\text{if } p \in M_{\text{int}}), \\ H_p^{\text{cap}} &:= \{\xi \in \Sigma_p \mid \xi \in \varphi(x)'_p, x \in \partial X_\nu, \sigma/10 \leq d(f(p), x) \leq \sigma\}, \\ &\quad (\text{if } p \in M_{\text{cap}}). \end{aligned}$$

which can be regarded as the sets of "horizontal directions" at  $p$ .

For  $\bar{p} := f(p) \in X_\nu$ , we also put

$$\begin{aligned} H_{\bar{p}}^{\text{int}} &:= \{\xi \in \Sigma_{\bar{p}} \mid \xi \in x'_{\bar{p}}, d(\bar{p}, \bar{x}) \geq \nu/10\}, \quad (\text{if } \bar{p} \in \partial X_\nu), \\ H_{\bar{p}}^{\text{cap}} &:= \{\xi \in \Sigma_{\bar{p}} \mid \xi \in x'_{\bar{p}}, \sigma/10 \leq d(\bar{p}, \bar{x}) \leq \sigma\}, \quad (\text{if } \bar{p} \in \partial X_\nu). \end{aligned}$$

For any  $\bar{\xi} \in H_{\bar{p}}^{\text{int}}$ , take  $\bar{q} \in X$  with  $\bar{\xi} = q'_{\bar{p}}$  and  $d(\bar{p}, \bar{q}) \geq \nu/10$ . Put  $\xi := (\varphi(\bar{q}))'_p \in H_p^{\text{int}}$ , and consider the map  $\chi_{\text{int}} : H_{\bar{p}}^{\text{int}} \rightarrow H_p^{\text{int}}$  defined by  $\chi_{\text{int}}(\bar{\xi}) = \xi$ . For any  $\bar{\xi} \in H_{\bar{p}}^{\text{cap}}$ , take  $\bar{q} \in \partial X$  with  $\bar{\xi} = q'_{\bar{p}}$  and  $\sigma/10 \leq d(\bar{p}, \bar{q}) \leq \sigma/10$ . Put  $\xi := (\varphi(\bar{q}))'_p \in H_p^{\text{cap}}$ , and consider the map  $\chi_{\text{cap}} : H_{\bar{p}}^{\text{cap}} \rightarrow H_p^{\text{cap}}$  defined by  $\chi_{\text{cap}}(\bar{\xi}) = \xi$ .

**Lemma 21.4.** *For sufficiently small  $t > 0$ , the following holds:*

- (1) *For every  $\bar{\xi} \in H_{\bar{p}}^{\text{int}}$ , put  $\xi := \chi_{\text{int}}(\bar{\xi})$ . Then*

$$d(f(\exp t\xi), \exp t\bar{\xi}) < t\tau(\delta, \sigma, \sigma_1/\sigma, \epsilon/\nu);$$

- (2) *For every  $\bar{\xi} \in H_{\bar{p}}^{\text{cap}}$ , put  $\xi := \chi_{\text{cap}}(\bar{\xi})$ . Then*

$$d(f(\exp t\xi), \exp t\bar{\xi}) < t\tau(\delta, \sigma, \sigma_1/\sigma, \nu/\sigma, \epsilon/\nu).$$



Conversely for every  $\xi \in H_p^{\text{int}}$  (resp.  $\xi \in H_p^{\text{cap}}$ ), there exists  $\bar{\xi} \in H_{\bar{p}}^{\text{int}}$  (resp.  $\bar{\xi} \in H_{\bar{p}}^{\text{cap}}$ ) satisfying the above inequality (1) (resp. (2)).

In other words, the curve  $f(\exp t\xi)$  is almost tangent to  $\exp t\bar{\xi}$ .

*Proof.* We prove (1). The proof of (2) is similar and hence omitted. By using (19.2), (21.1) and Lemma 21.3 we get  $|df_M(\xi) - df_X(\bar{\xi})| < \tau(\delta, \sigma, \epsilon/\nu)$ . Lemmas 20.4 and 21.1 then imply

$$\left| \frac{f_M(c(t)) - f_M(p)}{t} - \frac{f_X(\bar{c}(t)) - f_X(\bar{p})}{t} \right| < \tau(\delta, \sigma, \epsilon/\nu),$$

for sufficiently small  $t > 0$ . In particular  $f_M(c(t)) - f_M(p)$  is almost perpendicular to  $N_{\pi(f_M(p))}$ . It follows from 20.9 that

$$\left| \frac{f_M(c(t)) - f_M(p)}{t} - \frac{\pi \circ f_M(c(t)) - \pi \circ f_M(p)}{t} \right| < \tau(\delta, \sigma, \sigma_1/\sigma, \epsilon/\nu)$$

and hence  $|\pi \circ f_M(c(t)) - f_X(\bar{c}(t))| < t\tau(\delta, \sigma, \sigma_1/\sigma, \epsilon/\nu)$ . Lemma 19.2 then implies the required inequality.  $\square$

From now on we use the simpler notation  $\tau_{\delta, \nu, \epsilon}$  to denote a positive function of type  $\tau(\delta, \sigma, \sigma_1/\sigma, \nu/\sigma, \epsilon/\nu)$ .

We show that both  $f|_{M_{\text{int}}}$  and  $f|_{M_{\text{cap}}}$  are  $\tau_{\delta, \nu, \epsilon}$ -Lipschitz submersions.

The following fact follows from Lemma 21.4.

$$(21.3) \quad \left| \frac{d(f(\exp t\xi), \bar{p})}{t} - 1 \right| < \tau_{\delta, \nu, \epsilon},$$

for all  $\xi \in H_p^{\text{int}} \cup H_p^{\text{cap}}$  and sufficiently small  $t > 0$ .

**Lemma 21.5.** *For every  $p, q \in M_{\text{int}}$  (resp.  $p, q \in M_{\text{cap}}$ ), we have*

$$\left| \frac{d(f(p), f(q))}{d(p, q)} - \cos \theta \right| < \tau_{\delta, \nu, \epsilon},$$

where  $\theta = \angle(q'_p, H_p^{\text{int}})$  (resp.  $\theta = \angle(q'_p, H_p^{\text{cap}})$ ).

For the proof of Lemma 21.5 we need two sublemmas.

Lemma 21.3 and the second half of Lemma 21.4 imply the following

**Sublemma 21.6.**  $\chi_{\text{int}}$  (resp.  $\chi_{\text{cap}}$ ) gives a  $\tau(\delta, \sigma, \epsilon/\nu)$ -approximation (resp.  $\tau(\delta, \sigma, \nu/\sigma)$ -approximation).

In particular,  $d_{GH}(H_p^{\text{int}}, S^{k-1}(1)) < \tau_{\delta, \nu, \epsilon}$  and  $d_{GH}(H_p^{\text{cap}}, S^{k-2}(1)) < \tau_{\delta, \nu, \epsilon}$ .

Let  $H_p$  denote either  $H_p^{\text{int}}$  or  $H_p^{\text{cap}}$ .

**Sublemma 21.7.** *For any  $\xi \in \Sigma'_p$ , put  $\theta := \angle(\xi, H_p)$  and let  $\xi_1 \in H_p$  satisfy  $\theta = \angle(\xi, \xi_1)$ . Then*

$$d(f(\exp t\xi), f(\exp t \cos \theta \xi_1)) < t\tau_{\delta, \nu, \epsilon},$$

for every sufficiently small  $t > 0$ .

*Proof.* Since  $\Sigma_p$  has curvature  $\geq 1$ , we have an expanding map  $\rho : \Sigma_p \rightarrow S^{n-1}(1)$ , ( $n = \dim M$ ). First we show that  $|d(\rho(v_1), \rho(v_2)) - \angle(v_1, v_2)| < \tau_{\delta, \nu, \epsilon}$  for every  $v_1, v_2 \in H_p$ . Let  $v_1^* \in H_p$  be such that  $\angle(v_1, v_1^*) > \pi - \tau_{\delta, \nu, \epsilon}$ . Since  $\rho$  is expanding, we obtain that

$$(21.4) \quad |\angle(v_1, v_2) - d(\rho(v_1), \rho(v_2))| < \tau_{\delta, \nu, \epsilon}, \quad |\angle(v_1^*, v_2) - d(\rho(v_1^*), \rho(v_2))| < \tau_{\delta, \nu, \epsilon}.$$

Now we assume  $H_p = H_p^{\text{int}}$ . The case  $H_p = H_p^{\text{cap}}$  is similar and hence omitted. The above argument also implies that  $\rho(H_p)$  is  $\tau_{\delta, \nu, \epsilon}$ -Gromov-Hausdorff close to a totally geodesic  $(k-1)$ -sphere  $S^{k-1}(1)$  in  $S^{n-1}(1)$ . Let  $\zeta : H_p \rightarrow S^{k-1}(1) \subset S^{n-1}(1)$  be a  $\tau_{\delta, \nu, \epsilon}$ -approximation such that  $d(\zeta(v), \rho(v)) < \tau_{\delta, \nu, \epsilon}$  for all  $v \in H_p$ . For a given  $\xi \in \Sigma_p$ , an argument similar to (21.4) implies that  $|\angle(\xi, v) - d(\rho(\xi), \zeta(v))| < \tau$  for all  $v \in H_p$ . Remark that for any  $y$  with  $\sigma/10 \leq d(p, y) \leq \sigma$ , an elementary geometry yields

$$\cos d(\rho(\xi), \zeta(y'_p)) = \cos d(\rho(\xi), \eta) \cos d(\eta, \zeta(y'_p)),$$

where  $\eta$  is an element of  $S^{k-1}(1)$  such that  $\angle(\rho(\xi), \eta) = \angle(\rho(\xi), S^{k-1}(1))$ . It follows that for sufficiently small  $t > 0$

$$\begin{aligned} & |f_M(\exp t\xi) - f_M(\exp t \cos \theta \xi_1)|^2 / t^2 \\ &= \frac{\sigma^2}{b(\sigma)} \int_X \left( \frac{h(d(\exp t\xi, \varphi(x))) - h(d(p, \varphi(x)))}{t} \right. \\ &\quad \left. - \frac{h(d(\exp t \cos \theta \xi_1, \varphi(x))) - h(d(p, \varphi(x)))}{t} \right)^2 d\mu(x) \\ &\leq \frac{\sigma^2}{b(\sigma)} \int_X (h'(d(p, \varphi(x))))^2 (\cos \angle(\xi, \varphi(x)'_p) - \\ &\quad \cos \theta \cos \angle(\xi_1, \varphi(x)'_p))^2 d\mu(x) + \tau_{\delta, \nu, \epsilon} \\ &\leq \frac{\sigma^2}{b(\sigma)} \int_X (h')^2 (\cos \angle(\xi, \varphi(x)'_p) - \cos \angle(\rho(\xi), \rho(\varphi(x)'_p)) \\ &\quad + \cos \angle(\rho(\xi), \eta) \cos \angle(\eta, \rho(\varphi(x)'_p)) - \\ &\quad \cos \angle(\xi, \xi_1) \cos \angle(\xi_1, \varphi(x)'_p))^2 d\mu(x) + \tau_{\delta, \nu, \epsilon} \\ &\leq \tau_{\delta, \nu, \epsilon}. \end{aligned}$$

Therefore by Lemmas 20.9 and 19.2 we conclude the proof of the sub-lemma.  $\square$

*Proof of Lemma 21.5.* Suppose  $p, q \in M_{\text{int}}$ . Since  $f_{\text{int}}$  is a  $\tau(\epsilon)$ -approximation, we may assume that  $d(p, q) < \nu^2 \ll \nu$ . Let  $c : [0, \ell] \rightarrow M$  be a minimal geodesic joining  $p$  to  $q$  where  $\ell = d(p, q)$ . By using Lemma 18.6, one can show that

$$|\angle qc(t)x - \angle qp x| < \tau_{\delta, \nu, \epsilon},$$

for every  $t < \ell$  and for every  $x \in M$  with  $\nu/10 \leq d(p, x) \leq \nu$ . Let  $\xi_t := q'_{c(t)}$ , and  $\eta_0 \in H_p^{\text{int}}$  be such that  $\angle(\xi_0, H_{c(t)}^{\text{int}}) = \angle(\xi_0, \eta_0)$ . Take

a point  $y$  such that  $\eta_0 = y'_p$ ,  $\nu/10 \leq |py| \leq \nu$ . (21) implies that  $|\angle(y'_{c(t)}, \xi_t) - \angle(\xi_t, H_{c(t)}^{\text{int}})| < \tau_{\delta, \nu, \epsilon}$ . Put  $\theta_t = \angle qc(t)y$ . It follows from Sublemma 21.7 and (21) that

$$(21.5) \quad d(f \circ c(t+s), f(\exp s \cos \theta_0 \eta_t)) < \tau_{\delta, \nu, \epsilon} s,$$

where  $\eta_t := y'_{c(t)}$ . Put  $\bar{c}(t) = f \circ c(t)$ , and take any  $\bar{\eta}_t$  in  $\psi(y)'_{\bar{c}(t)}$ . Then by Lemma 21.4

$$(21.6) \quad d(f(\exp s \cos \theta_0 \eta_t), \exp s \cos \theta_0 \bar{\eta}_t) < \tau_{\delta, \nu, \epsilon} s.$$

By Lemma 18.6, we see that for every  $z \in X$  with  $\nu/10 \leq d(\bar{p}, z) \leq \nu$ ,

$$(21.7) \quad |\angle \psi(y) \bar{c}(t) z - \angle \psi(y) \bar{p} z| < \tau_{\delta, \nu, \epsilon}.$$

Now let  $(a_i, b_i)$  be a  $(k, \delta)$ -strainer at  $\bar{p}$  with  $d(\bar{p}, a_i) = \nu$  and  $\lambda : B_{\bar{p}}(\nu^2) \rightarrow \mathbb{R}^k$  be the bi-Lipschitz map defined by

$$\lambda(x) = (d(a_1, x), \dots, d(a_k, x)).$$

Put  $u(t) = \lambda \circ \bar{c}(t)$ . Combining (21.5), (21.6) and (21.7), we get

$$|\dot{u}(s) - \dot{u}(t)| < \tau_{\delta, \nu, \epsilon}, \quad ||\dot{u}(s)| - \cos \theta_0| < \tau_{\delta, \nu, \epsilon}.$$

for almost all  $s, t \in [0, \ell]$ . Thus we arrive at

$$\begin{aligned} & |\ell \dot{u}(s) - (\lambda(f(q)) - \lambda(f(p)))| \\ & \leq \int_0^\ell |\dot{u}(s) - \dot{u}(t)| dt \leq \tau_{\delta, \nu, \epsilon} \ell, \end{aligned}$$

from which the conclusion follows.

Next suppose  $p, q \in M_{\text{cap}}$ . We may assume  $d(p, q) < \sigma^2 \ll \sigma$ , since  $f_{\text{cap}}$  is a  $\tau(\nu)$ -approximation,

By using Lemma 18.7, we have

$$|\angle qc(t)\varphi(x) - \angle qp\varphi(x)| < \tau_{\delta, \nu, \epsilon},$$

for every  $t < \ell$  and for every  $x \in \partial X_\nu$  with  $\sigma/10 \leq d(p, x) \leq \sigma$ . Let  $\xi_t := q'_{c(t)}$ , and  $\eta_0 \in H_p^{\text{cap}}$  be such that  $\angle(\xi_0, H_p^{\text{cap}}) = \angle(\xi_0, \eta_0)$ . Take a point  $y \in \partial X_\nu$  such that  $\eta_0 = \varphi(y)'_p$ ,  $\sigma/10 \leq d(\bar{p}, y) \leq \sigma$ . By an argument similar to the previous one using a  $(k, \delta)$ -strainer  $(a_i, b_i)$  of  $D(X)$  at  $\bar{p}$  with  $d(\bar{p}, a_i) = \sigma$ , we obtain the required estimate.  $\square$

**Lemma 21.8.** *Let  $p \in M$  and  $x \in X_\nu$  be given.*

- (1) *If  $p \in M_{\text{int}}$ , then there exists a point  $q \in M_{\text{int}}$  such that  $f(q) = x$  and  $d(f(p), f(q)) \geq (1 - \tau_{\delta, \nu, \epsilon})d(p, q)$ .*
- (2) *If  $p \in M_{\text{cap}}$  and  $x \in \partial X_\nu$ , then there exists a point  $q \in M_{\text{cap}}$  such that  $f(q) = x$  and  $d(f(p), f(q)) \geq (1 - \tau_{\delta, \nu, \epsilon})d(p, q)$ .*

Namely  $f_{\text{int}}$  and  $f_{\text{cap}}$  are  $(1 - \tau_{\delta, \nu, \epsilon})$ -open in the sense of [BGP].

In view of Lemma 18.8, the proof of the lemma above is similar to [47] and hence omitted.

We are in a position to complete the proof of Theorem 18.4. So far we do not need the assumption that  $S_{\delta_n}(M)$  is empty. Now we use

this assumption to prove that both  $f_{\text{int}}$  and  $f_{\text{cap}}$  are locally trivial fibre bundle maps.

For any  $p \in M_{\text{int}}$ , set  $F := f_{\text{int}}^{-1}(\bar{p})$ , and take an  $(n, \delta)$ -strainer  $(a_i, b_i)$  at  $p$  such that  $(a_i)'_p, (b_i)'_p \in H_p^{\text{int}}$  for all  $1 \leq i \leq k$ . Note that  $(a_i)'_p$  and  $(b_i)'_p$  are almost tangent to  $F$  for  $k+1 \leq i \leq n$ . This implies that the map  $\Phi = (f, d_{a_{k+1}}, \dots, d_{a_n})$  is a bi-Lipschitz homeomorphism of a small neighborhood of  $p$  onto an open subset of  $X \times \mathbb{R}^{n-k}$ . It follows that  $f_{\text{int}}$  is a topological submersion and hence is a locally trivial bundle map by [44]. The proof for  $f_{\text{cap}}$  is similar and hence omitted.

*Proof of Corollary 1.4.* Since  $f_{\text{cap}} : N_{\text{cap}} \rightarrow \partial_0 Y_\nu$  is a locally trivial fibre bundle, the conclusion (1) follows from the generalized Margulis lemma of [20]. Note that  $f_{\text{cap}}$  is also an almost Lipschitz submersion. Therefore by the parametrized versions of Theorem 0.5 in [43] and Theorem 0.9 in the present paper, we can get the conclusions (2) and (3) respectively. Note that in the present case, we can prove the rescaling theorem corresponding to Theorem 4.1 by using the (generalized) argument of Lemma 4.8. The actual proofs of (2) and (3) are done by contradiction, and the details are omitted.  $\square$

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FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, FUKUOKA 812-8581, JAPAN

*Current address:* Institute of Mathematics, University of Tsukuba, Tsukuba 305-8571, JAPAN

*E-mail address:* takao@math.tsukuba.ac.jp